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ANALYSIS OF REGULARIZED NAVIER-STOKES

EQUATIONS - II

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ABSTRACT

A practically important regularization of the Navier-Stokes equations have been analyzed. As a continuation of the previous work, we study in this paper the structure of the attractors characterizing the solutions. Local as well as global invariant manifolds have been found. Regularity properties of these manifolds are analyzed.

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Chapter 1

Introduction

In [15] we have presented an analysis of a practically important form of the regularized Navier-Stokes equations. We also presented a theorem for the convergence of the solution of such system to the solution of the conventional Navier-Stokes system as the regularization parameter approaches zero.

In this paper we will present an analysis of the structure of the attractors associated with the regularized system. We will present in particular theorems for stable and unstable manifolds associated with each periodic solution and establish their analyticity and invariance properties. The main machinery needed for these invariant manifold theorems are the analyticity properties of the nonlinear semigroup and its Fréchet derivative and spectral theorems for the monodromy operator. These results are established in section 3.

In the section 4 of the paper we establish the existence of a global attractor for the system and prove its compactness. We also note certain bounds on the attractor which are uniform on the size of the regularization parameter. We then prove the existence of global (inertial) invariant varieties containing this attractor. Such a global invariant manifold theory is proposed in [6] for certain class of semilinear evolution equations. Motivation for such study of course comes from the famous paper of E. Hopf [8]. We then study the regularity of the inertial manifolds and obtain sufficient condition for them to be C^1.

Chapter 2

Governing Equations and Functional Framework

In this chapter we will briefly outline the mathematical framework used in this paper. For detail proofs of the relevant theorems see [15]. We regularize the conventional Navier-Stokes equations by adding a fourth order operator (Laplacian square) with an artificial dissipation parameter ϵ. In addition to the prescribed initial field u_0 and Dirichlet boundary condition, we also prescribe the Laplacian of the velocity field at the boundary to be zero. Let $\Omega \subset R^n, n \leq 6$ be a bounded open set of class $C^r, r \geq 4$. The problem is to find $(u, p) : \Omega \times (0, \infty) \to R^n \times R$ such that

$$\frac{\partial u}{\partial t} + \epsilon \Delta^2 u - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f, \qquad \text{in} \quad \Omega \times (0, \infty), \qquad (2.1)$$

$$\nabla \cdot u = 0, \qquad \text{in} \quad \Omega \times (0, \infty), \qquad (2.2)$$

$$u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \qquad (2.3)$$

$$u(x, 0) = u_0, \qquad \text{in} \quad \Omega. \qquad (2.4)$$

Here $\nu > 0$ is the coefficient of the kinematic viscosity of the fluid and f is a prescribed vectorfield.

Let us introduce the following function spaces:

$$j(\Omega) = \{u : \Omega \to R^n; \ u \in C^4(\Omega), \ u|_{\partial\Omega} = 0, \ \Delta u|_{\partial\Omega} = 0, \ \text{div } u = 0\},$$

$$H = \{u : \Omega \to R^n; \ u \in L^2(\Omega), \ \text{div } u = 0 \ u \cdot n|_{\partial\Omega} = 0\},$$

$$V = \{u : \Omega \to R^n; \ u \in H^2(\Omega), \ \text{div } u = 0, \ u|_{\partial\Omega} = 0\}.$$

Here we denote by $H^m(\Omega)$, the Hilbertian Sobolev space of (square integrable) vector-fields whose distributional derivatives up to order m are square integrable. These spaces are endowed with the inner product

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}$$

and the norm

$$\|u\|_{H^m(\Omega)} = \Big(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)} \Big)^{1/2} .$$

It can be shown [15] that the spaces H and V are respectively the completion of $j(\Omega)$ in the norm $L^2(\Omega)$ and $H^2(\Omega)$.

The space H is endowed with the inner product $(u, v)_{L^2}$ and norm $|u| = (u, u)^{1/2}_{L^2}$. One can easily verify that the norm induced by $H^2(\Omega)$ and the norm $\|\Delta u\|_{L^2(\Omega)}$ are equivalent in V. We then denote $\|u\| = \|\Delta u\|_{L^2(\Omega)} = (u, u)^{1/2}_V$ as the norm in V derived from the inner product

$$(u, v)_V = \sum_{i=1}^{n} \Big(\frac{\partial^2 u}{\partial x_i \partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_i} \Big) .$$

Let us now characterize a linear self adjoint positive operator A (which we call the dissipation operator) using the following fundamental linear problem : Find $(u, p) : \Omega \to R^n \times R$ such that

$$\begin{cases} \Delta^2 u + \nabla p = f, & \text{in } \Omega, \\ \nabla \cdot u = 0, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ \Delta u|_{\partial\Omega} = 0. \end{cases} \tag{2.5}$$

This equation characterizes a linear operator relating u to f and is defined on $j(\Omega)$. Let us denote the Friedrich's extension of this operator as A. This operator can be defined as follows: We define a positive definite, V-elliptic symmetric bilinear form $a(\cdot, \cdot) : V \times V \to R$ by

$$a(u, v) = \sum_{i=1}^{n} \Big(\frac{\partial^2 u}{\partial x_i \partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_i} \Big) .$$

Then by Lax-Milgram lemma we obtain an isometry $A \in \mathcal{L}(V; V')$ as

$$< Au, v >_{V' \times V} = a(u, v) = < f, v >_{V' \times V}, \quad \forall v \in V,$$

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and $Au = f \in V' = \mathcal{L}(V; R)$. We then define $D(A)$ as follows: for $f \in H \subset V', \exists\, u \in V$ such that

$$a(u, v) = (f, v)_H, \quad \forall\, v \in V.$$

We then denote $u \in D(A)$. We thus have $A \in \mathcal{L}(D(A); H) \cap \mathcal{L}(V; V')$. The operator A defined above is closed with $D(A)$ dense in $V \subset H$. From this it is easy to conclude that A is self adjoint. Since the continuous form $a(\cdot, \cdot)$ is positive definite we deduce from a theorem of Lions [13] that $D(A^{1/2}) = V$ and

$$a(u, v) = (A^{1/2}u, A^{1/2}v) \quad \forall\, u, v \in V.$$

This implies,

$$\|u\|^2 = a(u, u) = |A^{1/2}u|^2 \quad \forall\, u \in V.$$

We have in fact $A = A_1 A_2 : u \to f$ so that A is an isomorphism from $D(A)$ onto H with A_1 the Stokes operator and A_2 the Friedrich's extension of the Laplacian operator[15]. This gives

$$u \in D(A) = \{u \in H^4(\Omega); u|_{\partial\Omega} = 0, \Delta u|_{\partial\Omega} = 0, \text{ div} u = 0\}.$$

We also have the following estimate for solution (u, p)

$$\|u\|_{H^4(\Omega)} + \|p\|_{H^1(\Omega)/R} \leq c_0 \|f\|_H.$$

In consequence of the relationship $Au = f$, $\exists \beta_1, \beta_2 \in R^+$ such that

$$\beta_1 \|u\|_{H^4(\Omega)} \leq |Au| \leq \beta_2 \|u\|_{H^4(\Omega)}, \quad \forall\, u \in D(A). \tag{2.6}$$

By Rellich's Lemma [1] A^{-1} as a mapping in V' (or H) is compact. Hence the spectrum of operator A consists of real eigenvalues μ_j of finite multiplicities and can be ordered as

$$0 < \mu_1 \leq \mu_2 \leq \cdots, \qquad \mu_j \to +\infty \quad \text{as} \quad j \to +\infty,$$

with accumulation possible only at infinity. The self adjoint operator A possesses an orthonormal set of eigenfunctions $\{\phi_j\}_{j=1}^{\infty}$ complete in V' (or H).

$$A\phi_j = \mu_j \phi_j, \qquad \phi_j \in V(\text{or } D(A)), \forall\, j. \tag{2.7}$$

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If we identify H with its dual H' using the Riesz representation theorem then we get the following continuous dense and compact embedding structure :

$$D(A) \subset V \subset H \equiv H' \subset V' \subset D(A)'.$$

Let us now define the trilinear form $b(\cdot, \cdot, \cdot)$ associated with the inertia terms:

$$b(u, v, w) = \sum_{i,j=1}^{n} \int_{\Omega} u_i D_i v_j w_j \, dx, \qquad D_i = \frac{\partial}{\partial x_i}.$$

It can be easily shown by applying the divergence theorem and noting that v and w have zero trace,

$$b(u, v, w) = -b(u, w, v), \qquad \forall \, u, v, w \in V$$

and

$$b(u, v, v) = 0, \qquad \forall \, u, v \in V.$$

By the application of Hölder inequality and Sobolev embedding theorem we can show that $b(\cdot, \cdot, \cdot)$ is trilinear continuous on $H^{m_1}(\Omega) \times H^{m_2+1}(\Omega) \times H^{m_3}(\Omega)$, $m_i \geq 0$:

$$|b(u, v, w)| \leq c_0 \|u\|_{H^{m_1}(\Omega)} \|v\|_{H^{m_2+1}(\Omega)} \|w\|_{H^{m_3}(\Omega)},$$

$$\begin{aligned}
m_1 + m_2 + m_3 &\geq \frac{n}{2} \quad \text{if} \quad m_i \neq \frac{n}{2} \quad i = 1, 2, 3 \text{ and} \\
m_1 + m_2 + m_3 &> \frac{n}{2} \quad \text{if} \quad m_i = \frac{n}{2} \quad \text{for some } i \,.
\end{aligned} \tag{2.8}$$

In particular, for $n \leq 6$, b is a trilinear continuous form on $V \times V \times V$. When Ω is bounded, the following interpolation inequality holds [14]:

$$\|u\|_{H^{(1-\theta)m_1 + \theta m_2}} \leq c \|u\|_{H^{m_1}}^{(1-\theta)} \|u\|_{H^{m_2}}^{\theta}, \quad \forall \, u \in H^{m_2}(\Omega), \ m_1 \leq m_2, \ \theta \in]0, 1[. \tag{2.9}$$

From the above we will derive in particular,

$$|b(u, v, w)| \leq c_1 |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2} |w|, \quad \forall u \in V, v \in D(A), w \in H \tag{2.10}$$

$$|b(u, v, w)| \leq c_2 |u|^{1/2} \|u\|^{1/2} |v|^{1/2} \|v\|^{1/2} \|w\|, \qquad \forall \, u, v, w \in V, \tag{2.11}$$

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where c_1, c_2 are positive constants. By virtue of (2.8), we know that the above inequalities are valid for space dimension $n \leq 6$. The estimate (2.8) enables us to define (using Riesz representation theorem) a bilinear continuous operator B from $H^{m_1}(\Omega) \times H^{m_2+1}(\Omega)$ into $(H^{m_3}(\Omega))'$. In particular, for $u, v, w \in V$, $B(u, v) \in V'$ will be defined by

$$< B(u,v), w >_{V' \times V} = b(u,v,w), \quad \forall w \in V. \tag{2.12}$$

Let us note that a linear operator A_1 from V onto H can be defined as

$$(A_1 u, v) = (\nabla u, \nabla v), \quad \forall u, v \in V \tag{2.13}$$

and $A_1 u = -P_H \Delta u$, $\forall u \in V$. In fact A_1 is the Stokes operator associated with the conventional Navier-Stokes equations. P_H is the orthogonal projector in $L^2(\Omega)$ onto H.

Chapter 3

Local Invariant Manifolds

In this chapter we will establish the hyperbolicity of periodic solutions. Existence theorem for the periodic solutions is provided in [15]. Here we will study the orbits nearby each periodic orbit. We will prove in particular the existence, uniqueness and analyticity of stable and unstable manifolds. Such results for conventional Navier-Stokes equations have been proven in [20]. Let us consider a perturbation about a general time dependent smooth and bounded field $(U(x,t), P(x,t))$. We assume that this basic field satisfies the governing equations and the boundary conditions. Let us introduce $u = U + v$ and $p = P + q$ in (2.1)–(2.4). Then the perturbation solution (v, q) satisfies.

$$\begin{cases} \dfrac{\partial v}{\partial t} + \epsilon \Delta^2 v - \nu \Delta v + (U \cdot \nabla)v + (v \cdot \nabla)U + (v \cdot \nabla)v + \nabla q = 0, \\[2mm] \nabla \cdot v = 0, \qquad \text{in } \Omega \times (0, \infty), \\[2mm] v|_{\partial\Omega} = 0 \quad \Delta v|_{\partial\Omega} = 0, \\[2mm] v(x, 0) = v_0, \qquad \text{in } \Omega. \end{cases} \qquad (3.1)$$

Let us now rewrite this system as an equation of evolution in the Hilbert space V. This can be achieved by applying the projection operator P_H on the system (3.1). Noting that $P_H(\nabla q) = 0$, we get the evolution equation for v:

$$\begin{cases} \dfrac{dv}{dt} + \epsilon A v + \nu A_1 v + L_U(t)v + B(v, v) = 0, \quad t > 0, \\[2mm] v(0) = v_0 \in V. \end{cases} \qquad (3.2)$$

Here $A \in \mathcal{L}(V; V')$ is the dissipation operator defined earlier and $A_1 \in \mathcal{L}(V; H)$ is the

Stokes operator. We can use the Riesz representation theorem to characterize $L_U(t)$ and $B(\cdot, \cdot)$ as

$$< L_U(t)v, w >_{V' \times V} = b(U(t), v, w) + b(v, U(t), w), \quad \forall w \in V.$$

and

$$< B(v, v), w >_{V' \times V} = b(v, v, w), \quad \forall w \in V.$$

3.1 The Cauchy Problem and Associated Semigroup

In this section we will derive few useful properties of the semigroup generated by the dissipation operator $-\epsilon A$. These estimates will be used to resolve the nonlinear semigroup associated with the regularized Navier-Stokes equations as well as to establish certain results concerning the invariant manifolds.

Let us consider the Cauchy problem:

Problem 1 *Find $v \in C([0, \infty); V) \cap C^1(0, \infty; V')$ such that*

$$\begin{cases} \dfrac{dv}{dt} + \epsilon A v = 0, \quad t > 0, \\ v(0) = v_0 \in V. \end{cases}$$

Before studying the semigroup $S(t) : v_0 \to v(t)$ associated to the above problem, we will document certain relevant properties of the resolvent of $-\epsilon A$.

Lemma 3.1 *Let B_1 and B_2 be two Hilbert spaces defined below such that the embedding $B_1 \subset B_2$ is continuous and dense. Then the resolvent of the dissipation operator $-\epsilon A$ satisfies :*

$$\text{(i)} \quad \|R(\lambda; -\epsilon A)\|_{\mathcal{L}(B_2; B_2)} \leq \frac{1}{|\lambda_I|}, \quad \text{for } \lambda_I \neq 0, \text{ where } \lambda_I = Im\,\lambda, \quad (3.3)$$

$$\text{(ii)} \quad \|R(\lambda; -\epsilon A)\|_{\mathcal{L}(B_2; B_2)} \leq \frac{1}{\lambda_R + \epsilon \mu_1}, \quad \text{for } \lambda_R > -\epsilon \mu_1, \quad (3.4)$$

$$\text{(iii)} \quad \|R(\lambda; -\epsilon A)\|_{\mathcal{L}(B_2; B_2)} \leq \frac{M}{|\lambda + \epsilon a|}, \quad \text{for } \lambda \in \Sigma_\delta, \ \lambda \neq -\epsilon a, \quad (3.5)$$

where $\Sigma_\delta = \{\lambda; |arg(\lambda + \epsilon a)| \leq \pi/2 + \delta, 0 < \delta < \pi/2\}$ and $0 < a < \mu_1, \mu_1$ is the smallest eigenvalue of the operator A. Here we will take the spaces B_1 and B_2 as either

(a) $B_1 = V, \qquad B_2 = V'$

or

(b) $B_1 = D(A), \qquad B_2 = H.$

Proof: (i) let us write u in B_1 as an expansion of the eigenfunctions of A:

$$u = \sum_{k=1}^{\infty} (u, \phi_k)_{B_2} \phi_k \, .$$

Since ϕ_k are orthonormal,

$$\|R(\lambda; -\epsilon A)u\|_{B_2}^2 = \sum_{k=1}^{\infty} \frac{|(u, \phi_k)_{B_2}|^2}{|\lambda + \epsilon \mu_k|^2} \leq \frac{1}{|\lambda_I|^2} \|u\|_{B_2}^2 \, . \tag{3.6}$$

Here we used the fact that

$$\frac{1}{(\lambda_R + \epsilon \mu_k)^2 + |\lambda_I|^2} \leq \frac{1}{|\lambda_I|^2} \, , \qquad \forall k, \quad \lambda = \lambda_R + i \lambda_I \, .$$

Thus, the resolvent of $-\epsilon A$ in B_2 satisfies

$$\|R(\lambda; -\epsilon A)\|_{\mathcal{L}(B_2; B_2)} \leq \frac{1}{|\lambda_I|} \, , \qquad for \ \lambda_I \neq 0 \, .$$

(ii) Similarly, it follows from

$$\frac{1}{(\lambda_R + \epsilon \mu_k)^2 + |\lambda_I|^2} \leq \frac{1}{(\lambda_R + \epsilon \mu_1)^2} \, , \qquad \forall k,$$

that for $\lambda_R > -\epsilon \mu_1$,

$$\|R(\lambda; -\epsilon A)\|_{\mathcal{L}(B_2; B_2)} \leq \frac{1}{\lambda_R + \epsilon \mu_1} \, .$$

(iii) Combining (i) and (ii) we get for $0 < a < \mu_1$

$$\|R(\lambda; -\epsilon A)\|_{\mathcal{L}(B_2; B_2)} \leq \frac{\sqrt{2}}{|\lambda + \epsilon a|} \, , \quad \text{with} \quad \lambda_R > -\epsilon a \, .$$

From this we show by analytic continuation that

$$\|R(\lambda; -\epsilon A)\|_{\mathcal{L}(B_2; B_2)} \leq \frac{M}{|\lambda + \epsilon a|} \, , \quad for \ \lambda \in \Sigma_\delta, \ \lambda \neq -\epsilon a. \tag{3.7}$$

$\square$

Theorem 3.1 $-\epsilon A$ *generates a* C^0 *semigroup* $S(t)$ *such that:*

$$\text{(i)} \quad \|S(t)\|_{\mathcal{L}(B_2;B_2)} \leq e^{-\epsilon at}, \qquad t \geq 0,$$

$$\text{(ii)} \quad \|S(t)\|_{\mathcal{L}(B_2;B_1)} \leq \frac{c}{t}e^{-\epsilon at}, \qquad t > 0,$$

where $0 < a < \mu_1$ *and* c *depends on* ϵ. *Moreover* $S(t)$ *can be extended as a holomorphic semigroup in the sector* $\Delta_\delta = \{z : |arg\, z| < \delta, \delta = \tan^{-1}\frac{1}{ec}, Re\, z > 0\}$.

Proof: (i) The spectrum of the operator $-\epsilon A$ lies on the negative real axis, thus the resolvent set $\rho(-\epsilon A)$ will contain the positive real axis and from the estimate on the resolvent in Lemma 3.1, we have

$$\|R(\lambda; -\epsilon A)\|_{\mathcal{L}(B_2;B_2)} \leq \frac{1}{\lambda_R + \epsilon a} \qquad \text{for} \quad \lambda_R > -\epsilon a, \quad a < \mu_1.$$

Hence by the Hille-Yosida theorem [16], $-\epsilon A$ generates a strongly continuous semigroup $S(t)$ in B_2 and

$$\|S(t)\|_{\mathcal{L}(B_2;B_2)} \leq e^{-\epsilon at}, \qquad t \geq 0$$

(ii) Using the estimate (iii) of Lemma 3.1, We can represent $S(t)$ as an integral

$$S(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R(\lambda; -\epsilon A)\, d\lambda$$

where Γ is a smooth curve in $\sum_\delta$ consisting of two rays $xe^{i\theta}$ and $xe^{-i\theta}$, $0 < x < \infty$ and $\pi/2 < \theta < \pi$. Γ is oriented so that λ_I increasing along Γ. Differentiating the integral with respect to t, we get

$$S'(t) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t} R(\lambda; -\epsilon A)\, d\lambda\,.$$

From (3.5), for $\lambda \neq -\epsilon a$ and $t > 0$

$$\|S'(t)\|_{\mathcal{L}(B_2;B_2)} \leq \frac{Me^{-\epsilon at}}{\pi t}\int_0^\infty e^{-x\cos\theta}\, dx = \left(\frac{M}{\pi\cos\theta}\right)\frac{e^{-\epsilon at}}{t} = \frac{c}{t}e^{-\epsilon at}\,.$$

Consequently,

$$\epsilon\|S(t)\|_{\mathcal{L}(B_2;B_1)} = \epsilon\|AS(t)\|_{\mathcal{L}(B_2;B_2)} = \|S'(t)\|_{\mathcal{L}(B_2;B_2)} \leq \frac{c}{t}e^{-\epsilon at}, \quad \text{for } t > 0.$$

Now due to the differentiability of $S(t)$ for $t > 0$ and $S^{(n)}(t) = (-\epsilon A S(\frac{t}{n}))^n = (S'(\frac{t}{n}))^n$, we have

$$\frac{1}{n!}\|S^{(n)}(t)\|_{\mathcal{L}(B_2;B_2)} \leq \left(\frac{ce}{t}\right)^n e^{-\epsilon at}.$$

To extend the semigroup $S(z)$ in some sector, we consider power series for $S(z)$ around t

$$S(z) = S(t) + \sum_{n=1}^{\infty} \frac{S^{(n)}(t)}{n!}(z-t)^n.$$

This series will converge in $\mathcal{L}(B_2;B_2)$ for

$$|z - t| \leq k\left(\frac{t}{ce}\right), \quad \text{for } k < 1, \ t > 0.$$

Hence $S(t)$ can be extended to a holomorphic semigroup $S(z)$ in the sector $\Delta_\delta = \{z : |arg\, z| < \delta, \delta = \tan^{-1}\frac{1}{ce}, Re\, z > 0\}$.

$\square$

Using the estimate (iii) of Lemma 3.1, we can define the positive as well as negative powers of A. Since A^{-1} is compact, the spectral resolution of the self adjoint operator A can be used to define its fractional powers in a simple way. We thus write, $\forall \alpha \geq 0$

$$A^\alpha u = \sum_{k=1}^{\infty} \mu_k^\alpha (u, \phi_k)\phi_k, \quad \forall u \in D(A^\alpha),$$

with $D(A^\alpha) = \{u \in H \text{ such that } \sum_{k=1}^{\infty} \mu_k^{2\alpha}|(u, \phi_k)|^2 < \infty\}$.

Lemma 3.2 *For $\alpha > 0$ and $t > 0$, the bounded operator $A^\alpha S(t)$ satisfy the estimates:*

$$\|A^\alpha S(t)\|_{\mathcal{L}(B_2;B_2)} \leq \begin{cases} \dfrac{M}{t^\alpha} & \text{for } 0 < t < \dfrac{\alpha}{\epsilon\mu_1}, \\[2mm] \mu_1^\alpha e^{-\epsilon\mu_1 t} & \text{for } t \geq \dfrac{\alpha}{\epsilon\mu_1}, \end{cases}$$

where $M = (\frac{\alpha}{\epsilon e})^\alpha$.

Proof: Note that any element u in B_1 may be represented as

$$u = \sum_{k=1}^{\infty} (u, \phi_k)_{B_2}\phi_k.$$

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Hence we may write

$$A^\alpha S(t)u = \sum_{k=1}^\infty \mu_k^\alpha e^{-\epsilon\mu_k t}(u, \phi_k)_{B_2}\phi_k \,.$$

Thus,

$$\|A^\alpha S(t)\|_{\mathcal{L}(B_2;B_2)} \leq \max_{k\geq 1}(\mu_k^\alpha e^{-\epsilon\mu_k t}) \,.$$

Note that the right hand side of inequality admits maximum when $\mu_k = \dfrac{\alpha}{\epsilon t}$. This proves the lemma since $\mu_1 > 0$ is the smallest eigenvalue.

$\square$

Let P_N be an orthogonal projector in V' onto the finite dimensional subspace of span$\{\phi_1, \cdots, \phi_N\}$ and $Q_N = I - P_N$. Note that P_N and Q_N commute with A^α. The following lemma is useful in a later section and can be proved in a similar way to the previous lemma.

Lemma 3.3 *For $\alpha > 0$ and $t < 0$, the bounded operator $A^\alpha Q_N S(-t)$ satisfy the estimates:*

$$\|A^\alpha Q_N S(-t)\|_{\mathcal{L}(B_2;B_2)} \leq \begin{cases} \dfrac{M}{(-t)^\alpha} & \textit{for } -\dfrac{\alpha}{\epsilon\mu_{N+1}} \leq t < 0 \,, \\[4mm] \mu_{N+1}^\alpha e^{\epsilon\mu_{N+1}t} & \textit{for } -\infty < t \leq -\dfrac{\alpha}{\epsilon\mu_{N+1}}, \end{cases}$$

where $M = \left(\dfrac{\alpha}{\epsilon e}\right)^\alpha$.

$\square$

Since semigroup $S(t)$ is holomorphic we have for $t_2 \geq t_1 > 0$

$$S(t_2) - S(t_1) = \int_{t_1}^{t_2} \frac{dS(t)}{dt}dt = \int_{t_1}^{t_2}(-\epsilon A)S(t)dt.$$

This gives

$$\|S(t_2) - S(t_1)\|_{\mathcal{L}(H;V)} \leq \epsilon\int_{t_1}^{t_2}\|AS(t)\|_{\mathcal{L}(H;V)}dt = \epsilon\int_{t_1}^{t_2}\|A^{3/2}S(t)\|_{\mathcal{L}(H;H)}dt.$$

By applying Lemma 3.2 with $\alpha = 3/2$ and $B_2 = H$, we obtain

$$\|S(t_2) - S(t_1)\|_{\mathcal{L}(H;V)} \leq c_1\left(\frac{1}{t_1^{1/2}} - \frac{1}{t_2^{1/2}}\right), \qquad t_2 \geq t_1 > 0. \tag{3.8}$$

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3.2 The Characterization of the Monodromy Operator

We will now characterize the evolution operator $Z(\cdot, \cdot)$ associated with the Cauchy's problem obtained by linearizing the regularized Navier-Stokes equations about a smooth time dependent basic field. Let us consider the

Problem 2 *Find* $v \in C([0, \infty); V) \cap C^1((0, \infty); V')$ *such that*

$$\frac{dv}{dt} + \epsilon A v + \nu A_1 v + L_U(t)v = 0, \qquad t > 0, \tag{3.9}$$

$$v(0) = v_0 \in V .$$

We will show that the Problem 2 is equivalent to the following integral representation for $v(t)$:

Problem 3 *Find* $v \in C([0, \infty); V)$ *such that,*

$$v(t) = S(t)v_0 - \int_0^t S(t - \tau)[\nu A_1 + L_U(\tau)]v(\tau)d\tau , \tag{3.10}$$

$$v(0) = v_0 \in V .$$

Theorem 3.2 *Let the basic field satisfies* $U \in C([0, \infty); H^1(\Omega))$ *and be bounded. Then the Problem 3 resolves the Problem 2.*

Note that the Stokes operator A_1 is a linear continuous operator from V onto H. Furthermore, the linear operator $L_U(t)$ is characterized by

$$< L_U(t)v, w >_{V' \times V} = b(U(t), v, w) + b(v, U(t), w), \quad \forall w \in V.$$

By virtue of (2.8) we obtain the following lemma.

Lemma 3.4 *If* $U \in H^1(\Omega)$, $L_U \in \mathcal{L}(V; H)$.

Proof: Recalling the estimate for $b(\cdot, \cdot, \cdot)$ in (2.8), we take $m_1 = 2$ and $m_2 = m_3 = 0$ to get

$$|b(v, U, w)| \leq c_1 \|v\|_{H^2(\Omega)} \|U\|_{H^1(\Omega)} \|w\|_{L^2(\Omega)} , \quad \forall w \in H .$$

A similar estimate holds for $b(U, v, w)$. Thus

$$|(L_U v, w)_{L^2(\Omega)}| \leq (c_1 + c_2) \|v\|_V \|U\|_{H^1(\Omega)} \|w\|_{L^2(\Omega)} , \quad \forall w \in H .$$

Here we note that the norm induced by $H^2(\Omega)$ is equivalent to the norm in V. Setting $w = L_U v \in H$ we get,

$$\|L_U v\|_{L^2(\Omega)} \leq (c_1 + c_2) \|U\|_{H^1(\Omega)} \|v\|_V.$$

$\square$

In order to prove Theorem 3.2, we need first to establish certain properties of linear operator K defined as

$$[Kv](t) = -\int_0^t S(t - \tau)[\nu A_1 + L_U(\tau)]v(\tau)d\tau .$$

Lemma 3.5 *Let K be the linear operator defined above. Then for sufficiently small T_1, K is a contraction in $C([0, T_1]; V)$. Moreover, K can be extended as a contraction in the Banach space B defined as: $B = \{t \rightarrow v(t); v(t)$ continuous in V for $t \in]0, T_1]$ and $t^{1/2} v(t)$ bounded in $V\}$, with norm*

$$\|v\|_B = \sup_{t \in (0, T_1)} \|t^{1/2} v(t)\|_V .$$

$\square$

Lemma 3.6 *For $v \in C([0, T_1]; V)$, the time derivative $[Kv]'(\cdot) \in C((0, T_1); V')$. Moreover, when $v \in B$ the map $t \rightarrow [Kv]'(t)$ is continuous from $]0, T_1)$ into V'.*

The proofs for the above two lemmas are similar to those for the conventional Navier-Stokes equations [20]. Note that $[Kv](t) \in C([0, T_1]; V)$ and it has continuous right derivative $[Kv]'_+(t) \in C((0, T_1); V')$. This implies $[Kv](t)$ is strongly differentiable and $[Kv]'(t) = [Kv]'_+(t)$ for $t \in (0, T_1)$ (see Zaidman [23]).

Proof of Theorem 3.2: We have shown that the strong derivative $[\mathcal{K}v]'(t)$ exists and $[\mathcal{K}v]'(t) = -\epsilon A[\mathcal{K}v](t) - [\nu A_1 + L_U(t)]v(t)$. This gives

$$[\mathcal{K}v]'(t) + \epsilon A[\mathcal{K}v](t) + [\nu A_1 + L_U(t)]v(t) = 0\,,$$

with $[\mathcal{K}v](0) = 0$ which implies the representation (3.10) in Problem 3 satisfies the differential equation (3.9) in Problem 2. Moreover, due to the properties of the linear semigroup $S(t)$ established in previous section we have,

$$S(t)v_0 \in C([0,\infty); V) \cap C^1((0,\infty); V')\,.$$

From Lemma 3.5 and 3.6, we can conclude that

$$\int_0^t S(t-\tau)[\nu A_1 + L_U(\tau)]v(\tau)d\tau \in C([0,T_1); V) \cap C^1((0,T_1); V').$$

Let us now characterize the evolution operator associated to the linear differential equation in Problem 2. From the definition of linear operator $\mathcal{K}$, we can rewrite (3.10) in Problem 3 as

$$[(I-\mathcal{K})v)](t) = S(t)v_0.$$

Then for small enough T_1, the operator $(I - \mathcal{K})$ is invertable in $C([0,T_1); V)$ and in B since $\mathcal{K}$ is a contraction in these spaces. We obtain a convergent series in $C([0,T_1); V)$ as

$$v(\cdot) = \sum_{n=0}^{\infty} [\mathcal{K}^n S](\cdot)v_0. \tag{3.11}$$

Hence, the solution of Problem 3 can be denoted by $v(t) = Z(t,0)v_0$. We call $Z(t,0)$ the evolution operator. Note that the convergence of this series ensures the uniqueness of the solution to Problem 3 (and hence to the Problem 2.) Let us now study the evolution for $t \geq \tau$ by prescribing the initial data at $t = \tau$ in Problems 2 and 3. The evolution operator obtained(as a series) in this manner is denoted $Z(t,\tau)$ with $t - \tau \leq T_1$. Here T_1 is taken

small enough to ensure the convergence of the series. Let us consider for $0 \leq \eta \leq \tau \leq t$, $v(t) = Z(t,\tau)v(\tau)$ with $v(\tau) = Z(\tau,\eta)v_0$. That is

$$v(t) = Z(t,\tau)Z(\tau,\eta)v_0 \, .$$

Due to the uniqueness of the solution we have

$$v(t) = Z(t,\eta)v_0 = Z(t,\tau)Z(\tau,\eta)v_0 \, .$$

That is

$$Z(t,\eta) = Z(t,\tau)Z(\tau,\eta), \, 0 \leq \eta \leq \tau \leq t \, .$$

Iterating this kind of arguements we can extend the definition of $Z(t_2,t_1)$ to $t_2 - t_1 \in [0,\infty)$.

$\square$

The next series of lemmas provide useful regularity and compactness properties of evolution operator $Z(\cdot,\cdot)$. These lemmas can be proved using the same methods used in the context of conventional Navier-Stokes equations [20]. Note that here the initial data is prescribed in V for the evolution problem. This means we need to characterize $Z(\cdot,\cdot)$ as an element in $\mathcal{L}(V;V)$. Moreover, the bilinear operator characterizing the inertia term has the property $B(\cdot,\cdot) \in \mathcal{L}(V \times V; H)$. Hence we need to extend $Z(\cdot,\cdot)$ as an element in $\mathcal{L}(H;V)$.

Lemma 3.7 *For $0 \leq \tau < t$ the evolution operator $Z(t,\tau)$ satisfies the following estimates:*

(i) $\quad \|Z(t,\tau)\|_{\mathcal{L}(V;V)} \leq c_3 e^{\sigma_1(t-\tau)} \, ,$

(ii) $\quad \|Z(t,\tau)\|_{\mathcal{L}(H;V)} \leq c_4 \{1 + \dfrac{1}{(t-\tau)^{1/2}}\} e^{\sigma_2(t-\tau)} \, .$

Here $c_3, c_4 > 0$ and $\sigma_1, \sigma_2 \geq 0$.

$\square$

Lemma 3.8 *For $0 \leq t_1 \leq t_2 < \infty$ we have as $t_2 \to t_1^+, Z(t_2, t_1) \to I$ strongly in $\mathcal{L}(V; V)$. For $t > \tau$ the map $t \to Z(t, \tau)$ is continuous in the uniform operator topology of $\mathcal{L}(V; V) \cap \mathcal{L}(H; V)$ and for $\tau < t$ the map $\tau \to Z(t, \tau)$ is continuous in the uniform operator topology of $\mathcal{L}(V; V) \cap \mathcal{L}(H; V)$.*

$\square$

Lemma 3.9 *For $0 \leq \tau < t$ the operator $Z(t, \tau) \in \mathcal{L}(V; V) \cap \mathcal{L}(H; V)$ is compact.*

$\square$

Note that the evolution operator $Z(t_2, t_1) : H \to V$ is compact and hence the spectrum of this operator is discrete with finite multiplicity and accumulation possible only at the origin. Moreover, they are the same in H and V.

Let us now specialize our study to the case where the **basic field U is T-periodic in time.**

Lemma 3.10 *Let the basic field be T-periodic in time. Then*
(i) $Z(nT, 0) = Z(T, 0)^n$, $\forall n \geq 1$,
(ii) The spectrum of $Z(T + t_0, t_0)$ is independent of $t_0 \geq 0$.

$\square$

We will call the operator $Z(T, 0)$ the **Monodromy operator.**

3.3   The Nonlinear Semigroup

In this section we will characterize nonlinear semigroup associated with the nonlinear syatem (3.1). We will define in particular the time T-map which relate the initial data v_0 to the solution v at time T. In addition, we will prove that the dependence of v in the initial data is Fréchet analytic. This last result will be used in next section to establish the analyticity of the local invariant manifolds. Let us now consider the evolution form of the regularized Navier-Stokes system.

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Problem 4 *Find $v \in C([0,\infty); V) \cap C^1(0,\infty; V')$ such that*

$$\frac{dv}{dt} + \epsilon A v + \nu A_1 v + L_U(t) v + B(v,v) = 0, \quad t > 0, \tag{3.12}$$

$$v(0) = v_0 \in V .$$

Here the bilinear operator $B(\cdot, \cdot)$ is defined using the Riesz representation theorem as

$$< B(v, u), w >_{V' \times V} = b(v, u, w), \ \forall w \in V .$$

The evolution Problem 4 can be derived by simply applying the projection operator P_H onto the system (3.1).

Let us consider the following integral representation for the solution of (3.12):

Problem 5 *Find $v \in C([0,\infty); V)$ such that*

$$v(t) = Z(t,0) v_0 - \int_0^t Z(t,\eta) B(v(\eta), v(\eta)) d\eta , \tag{3.13}$$

$$v(0) = v_0 \in V ,$$

where $Z(\cdot, \cdot)$ is the evolution operator described in preceding section.

Theorem 3.3 *Problem 5 resolves Problem 4.*

Proof: First note that the representation (3.13) in Problem 5 formally satisfies the evolution form (3.12) in Problem 4. From the regularity properties of the evolution operator $Z(t,0)$ we have,

$$Z(t,0) v_0 \in C([0,\infty); V) \cap C^1(0,\infty; V')$$

Hence if we set,

$$Y(t) = - \int_0^t Z(t,\eta) B(v(\eta), v(\eta)) d\eta, \tag{3.14}$$

then we only need to show that $Y(\cdot) \in C([0,\infty); V) \cap C^1(0,\infty; V')$. Let us first show that $Y(t)$ is bounded and continuous in V.

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Note first that $B(\cdot, \cdot) \in \mathcal{L}(V \times V; H)$. In fact from the estimate for $b(\cdot, \cdot, \cdot)$ in (2.8) with $m_1 = 2, m_2 = 1, m_3 = 0$ we get

$$|b(u, v, w)| \leq c_0 \|u\|_{H^2(\Omega)} \|v\|_{H^2(\Omega)} \|w\|_{L^2(\Omega)}, \quad \forall u \in V, \ w \in H.$$

Hence by the Riesz representation theorem we can write

$$b(u, v, w) = (B(u, v), w)_{H \times H}, \forall w \in H.$$

By setting $w = B(u, v) \in H$, this gives

$$\|B(u, v)\|_H \leq c_0 \|u\|_V \|v\|_V.$$

Thus $B(v, v) \in C([0, \infty); H)$ for $v \in C([0, \infty); V)$.

Let us now estimate (3.14) as,

$$\|Y(t)\|_V \leq \int_0^t \|Z(t, \eta)\|_{\mathcal{L}(H; V)} \|B(v(\eta), v(\eta))\|_H d\eta.$$

Now using the estimates for the evolution operator $Z(t, \tau)$ obtained in the previous chapter we get,

$$\|Y\|_{C([0, T_2); V)} \leq y_1(T_2) \|v\|^2_{C([0, T_2); V)}.$$

Let us now consider,

$$Y(t + h) - Y(t) = -\int_0^t [Z(t + h, \eta) - Z(t, \eta)] B(v(\eta), v(\eta)) d\eta$$
$$- \int_t^{t+h} Z(t + h, \eta) B(v(\eta), v(\eta)) d\eta.$$

Estimating this we get,

$$\|Y(t + h) - Y(t)\|_V \leq \{\int_0^t \|Z(t + h, \eta) - Z(t, \eta)\|_{\mathcal{L}(H; V)} d\eta$$
$$+ \int_t^{t+h} \|Z(t + h, \eta)\|_{\mathcal{L}(H; V)} d\eta\} \|B(v, v)\|_{C(0, T_2; H)}.$$

Again using the results of the previous chapter we get

$$\|Y(t + h) - Y(t)\|_V \leq y_2(t, h) \|v\|^2_{C([0, T_2); V)}$$

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with $y_2(t, h) \to 0$ as $h \to 0$. We hence conclude that $Y \in C([0, T_2); V)$. Now,

$$\frac{Y(t + h) - Y(t)}{h} = -\left(\frac{Z(t + h, t) - Z(t, t)}{h}\right) \int_0^t Z(t, \eta) B(v(\eta), v(\eta)) d\eta$$
$$- Z(t + h, t) \frac{1}{h} \int_t^{t+h} Z(t, \eta) B(v(\eta), v(\eta)) d\eta .$$

Taking the limit $h \to 0$ we get due to the continuity results of the evolution operator $Z(\cdot, \cdot)$ obtained last chapter

$$Y_+'(t) = -[\epsilon A + \nu A_1 + L_U(t)] Y(t) - B(v(t), v(t)) .$$

Here $Y_+'(\cdot)$ is the right derivative.

For $v \in C([0, T_2); V)$ we have $B(v, v) \in C([0, T_2); H)$ and $Y \in C([0, T_2); V)$. Since A is an isomorphism from V onto V', we have $AY \in C([0, T_2); V')$ and $[A_1 + L_U(t)] Y \in C([0, T_2); H)$ due to the estimates on the trilinear form $b(\cdot, \cdot, \cdot)$. Hence we conclude that the right derivative exists and $Y_+'(\cdot) \in C(0, T_2; V')$. From this as before we conclude that the strong derivative $Y'(\cdot) \in C(0, T_2; V')$. Thus

$$Y'(t) + \epsilon A Y(t) + \nu A_1 Y(t) + L_U(t) Y(t) + B(v(t), v(t)) = 0,$$

which implies that $Y(t)$ is a solution of Problem 4 with $Y(0) = 0$. This proves the Theorem.

$\square$

The existence and uniqueness aspects of the solution can be established using the methods used in [9]. Here one shows that for a fixed time interval there exists a neighborhood such that there is a unique solution for each initial data in this neighborhood. The time interval can be taken to infinity by choosing this neighborhood sufficiently small. (However, see remark.)

We will now establish an important result regarding the dependence of the solution on the initial data. Let us rewrite Problem 5 in the form $\mathcal{F}(v, v_0) = 0$. Here the map

$$\mathcal{F}(\cdot, \cdot) : C(0, T_2; V) \times V \to C(0, T_2; V)$$

is defined by

$$\mathcal{F}(v, v_0) = v - Z(t,0)v_0 + M(v,v),\qquad (3.15)$$

$$\text{with} \quad M(v,v) = \int_0^t Z(t,\eta)B(v(\eta),v(\eta))d\eta.$$

Here the bilinear operator $M(\cdot,\cdot) : C(0,T_2;V)^{\otimes 2} \longrightarrow C(0,T_2;V)$ satisfies

$$M(0,w) = M(w,0) = 0, \quad \forall\, w \in C(0,T_2;V)$$

and $Z(t,0) \in \mathcal{L}(V,V)$ for fixed t.

The following theorem is a consequence of the analytic version of the implicit function theorem [5]. Results of this type for the conventional Navier-Stokes system is given in Sritharan [20].

Theorem 3.4 *For fixed $T^* > 0$ there exists a neighborhood of the origin $B_{2\delta} \subset V$ such that for $v_0 \in B_{2\delta}$, there exists a unique solution $v \in C(0,T^*;V)$ to the problem 5. Moreover, the dependence of v in the initial data v_0 is Fréchet-analytic.*

$\square$

This means there exists a map

$$W(t,0;\cdot) : B_{2\delta} \subset V \longrightarrow B_{1\delta} \subset C(0,T^*;V)$$

such that $v(t) = W(t,0;v_0)$ and $W(t,0;\cdot)$ is F-analytic in this neighborhood. That is $v(t)$ can be written as a power series in the initial data in the following way:

$$v(t) = W(t,0;v_0) = \sum_{n \geq 1} \mathcal{H}_n(v_0,\cdots,v_0;t).\qquad (3.16)$$

The n-linear maps

$$\mathcal{H}_n(\cdots) : B_{2\delta}^{\otimes n} \to B_{1\delta} \quad \text{continously}.$$

This series representation converges in the neighborhood defined above. One can verify easily that $\mathcal{H}_1(v_0;t) = Z(t,0)v_0$. We finally note that due to the uniqueness theorem,

$$W(t,0;v_0) = W(t,t_1;W(t_1,0;v_0)), \quad \text{for } 0 \leq t_1 \leq t < \infty,$$

and $W(0, 0; \cdot) = I$.

If the basic flow field $U(t)$ is T-periodic then

$$W(2T, 0; v_0) = W(T, 0; W(T, 0; v_0))$$

and in general

$$W(nT, 0; \cdot) = W(T, 0; \cdot)^n, \quad \text{for} \quad n \geq 1. \tag{3.17}$$

As noted earlier in the previous section, the Frechét derivative of the solution map $(DW)(t, 0; \cdot) = Z(t, 0)$ satisfies a similar relationship as in (3.17).

<u>Remark:</u> The nonlinear semigroup characterized above is defind (when $T^* = \infty$) only in a neighborhood of the origin. However in [15] we have proved the global existence and uniqueness of strong solution in V using other methods.

3.4   Local Invariant Manifold Theorem

Let the spectrum of the monodromy operator $Z(T, 0) \in \mathcal{L}(V; V)$ splits into two disjoint sets σ_U and σ_S such that $\sigma(Z(T, 0)) = \sigma_U \cup \sigma_S$ and

$$b_S = \sup_{\lambda \in \sigma_S} |\lambda| < \inf_{\lambda \in \sigma_U} |\lambda| = b_U^{-1}. \tag{3.18}$$

Let P_U and P_S be the spectral projectors defined by the Dunford's integrals,

$$P_U = \frac{1}{2\pi i} \int_{\Gamma_U} R(\lambda; Z(T, 0)) d\lambda$$

$$\text{and} \quad P_S = \frac{1}{2\pi i} \int_{\Gamma_S} R(\lambda; Z(T, 0)) d\lambda.$$

Here $R(\lambda; Z(T, 0))$ is the resolvent operator and Γ_U, Γ_S encircle σ_U, σ_S respectively. Note that $P_S + P_U = I$, $P_S P_U = P_U P_S$ and P_S, P_U commute with $Z(T, 0)$.

Theorem 3.5 (The invariant cone theorem) *Let the basic flow be T-periodic in time so that the solution map satisfies (3.17).*

(i) *If the spectrum of the monodromy operator $Z(T, 0)$ lies inside the unit disc (spectral radius < 1) then the basic periodic solution is (locally) exponentially stable: there exists*

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$\rho > 0$ such that $\forall \, v_0 \in B_\rho(0) \subset V$, $W(t,0;v_0) \to 0$ exponentially in the norm of V.

(ii) Let the spectrum of $Z(T,0)$ satisfy (3.18) with $b_U < 1$, then there exists a double cone $K \subset V$ and a ball $B_\delta(0) \subset V$ such that $\forall v_0 \in B_\delta(0) \cap K \backslash \{0\}$, there exists $n \in \mathcal{N}$ for which $\|W(nT,0;v_0)\|_V > \delta$. That is the basic solution is Lyapunov unstable. Here double cone is defined by

$$K = \{v \in V \text{ such that } \|P_S v\|_V \leq \gamma \|P_U v\|_V, \ \gamma > 0\}. \tag{3.19}$$

$\square$

Theorem 3.6 (The invariant manifold theorem) Let $b_S, b_U < 1$. Then in a neighborhood $B_r(0) \subset V$, there exists two unique, analytic manifolds M_S and M_U which are respectively the graphs of the maps, $\phi_S : P_S V \to P_U V$ and $\phi_U : P_U V \to P_S V$. The maps ϕ_S and ϕ_U are analytic with,

1. $\phi_U(0) = \phi_S(0) = 0$;

2. $D\phi_U(0) = D\phi_S(0) = 0$ (tangency condition;)

3. manifolds M_S and M_U are locally invariant under the solution map $W(T,0;\cdot)$

$$W(T,0;M_U \cap B_r(0)) \subset M_U \quad \text{and} \quad W(T,0;M_S \cap B_r(0)) \subset M_S;$$

4. stable manifold M_S satisfies

$$M_S \cap B_r(0) = \{v \in B_r(0) \text{ such that } \forall \, n \geq 0, W(nT,0;v) \in B_r(0)$$
$$\text{and} \to 0 \text{ as } n \to \infty\};$$

5. Unstable manifold M_U satisfies,

$$M_U \cap B_r(0) = \{v \in B_r(0) \text{ such that } W(T,0;\cdot)^n v \text{ is defined } \forall \, n < 0$$
$$\text{and tends to zero as } n \to -\infty\};$$

6. if $v \notin M_S$ then there exists $\delta > 0$ and $p \in N$ such that, $\|W(pT,0;v)\|_V > \delta$;

7. $\text{dist}(M_U, W(T, 0; v)) < \text{dist}(M_U, v)$ *for* $v \in B_r(0)$ *(exponential attractive property of the unstable manifold;)*

8. $\text{dist}(M_S, W(T, 0; v)) > \text{dist}(M_S, v)$ *for* $v \in B_r(0)$ *(repelling property of the stable manifold.)*

Proofs of Theorem 3.5 and 3.6 are similar to those for the conventional Navier-Stokes system and can be found in detail in [20].

$\square$

Chapter 4

Global Invariant Varieties (Inertial Varieties)

Foias, Sell and Temam proposed in [6] the concept of inertial varieties for certain class of semilinear evolution equations. In this section we will study the existence of such global invariant varieties modelled on the invariant subspaces of A. These manifolds will be invariant to the action of the $W(t, 0, \cdot)$. In section 4.5 we will extend the general theory in [6] to analyze the regularity of the inertial manifolds. Let us first obtain certain overall bounds for the solution in various norms and show that the dynamics is characterized by a compact global attractor. The global manifolds to be constructed will contain this attractor.

4.1 Overall Bounds for the Solutions

We can get a weak formulation from the system (2.1)-(2.4) by taking duality pairing with $w \in V$,

$$< \frac{\partial u}{\partial t}, w > + \epsilon(\Delta u, \Delta w) + \nu(\nabla u, \nabla w) + b(u, u, w) = < f, w > \quad \forall w \in V \qquad (4.1)$$

We will first consider the energy estimate by setting $w = u$ and using the fact that $b(u, u, u) = 0$,

$$\frac{1}{2} \frac{d}{dt} |u|^2 + \epsilon \|u\|^2 + \nu |\nabla u|^2 = < f, u > . \qquad (4.2)$$

Using the Poincaré's Lemma $|\nabla u|^2 \geq \eta_1 |u|^2$ and the fact $\|u\|^2 \geq \mu_1 |u|^2$ we get

$$\frac{d}{dt}|u|^2 + (\epsilon\mu_1 + 2\nu\eta_1)|u|^2 \leq \frac{\|f\|_{V'}^2}{\epsilon}.$$

Here we denote η_1 as the smallest eigenvalue of Stokes operator A_1. We then set $\alpha = \epsilon\mu_1 + 2\nu\eta_1$ and $\rho_0^2 = \frac{\|f\|_{V'}^2}{\epsilon\alpha}$ to obtain

$$|u(t)|^2 \leq |u(0)|^2 e^{-\alpha t} + \rho_0^2 \left[1 - e^{-\alpha t} \right], \quad \alpha > 0. \tag{4.3}$$

We note that in the case of $|u(x,0)| > \rho_0$, the energy estimate in (4.3) show that $|u(t)|$ is bounded by a monotone decreasing exponential funtion. On the other hand, if $|u(x,0)| \leq \rho_0$, then $|u(x,t)| \leq \rho_0$ for $\forall t \in R^+$. Hence for any ball $B_R = \{u(0) \in H; |u(0)| \leq R\}$, there is a ball B_{r_0} in H centered at origin with radius $r_0 > \rho_0$ such that

$$W(t,0;B_R) \subset B_{r_0}, \quad \text{for } t \geq t_0(B_R) = \frac{1}{\alpha} \ln \frac{R^2 - \rho_0^2}{r_0^2 - \rho_0^2}. \tag{4.4}$$

The ball B_{r_0} is said to be **exponentially absorbing** and **invariant** [10,11,6] under the action of the map $W(t,0;\cdot)$.

Let us now proceed to get other estimates. Notice that from energy estimate if $u_0 \in B_R$ and $t \geq t_0(B_R)$, by integrating (4.2) from t to $t+1$,

$$\int_t^{t+1} \|u\|^2 d\tau \leq \frac{1}{\epsilon} \left[r_0^2 + \frac{1}{\epsilon}\|f\|_{V'}^2 \right] = \sigma_0. \tag{4.5}$$

To prove uniform bounds on different norms we use the uniform Gronwall inequality [6,15]:

Lemma 4.1 (Uniform Gronwall Inequality) *Let g, h, y be three positive locally integrable functions for $t^* \leq t < +\infty$ which satisfy*

$$\frac{dy}{dt} \leq gy + h \qquad \forall\ t \geq t^* \text{ and}$$

$$\int_t^{t+1} g(s)ds \leq \alpha_1, \quad \int_t^{t+1} h(s)ds \leq \alpha_2, \quad \int_t^{t+1} y(s)ds \leq \alpha_3,$$

for all $t \geq t^*$, where $\alpha_1, \alpha_2, \alpha_3$ are positive constants. Then

$$y(t+1) \leq (\alpha_2 + \alpha_3)exp(\alpha_1), \qquad \forall\ t \geq t^*. \tag{4.6}$$

Let us now suppose that $f \in H$ and set $w = Au$ in (4.1),

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \epsilon|Au|^2 + \nu|\nabla A_2 u|^2 \le c_1|u|^{1/2}\|u\| |Au|^{3/2} + |f| |Au|.$$

Here we have used the estimate in (2.10) for trilinear form $b(u, u, Au)$. Using the Young's inequality, we obtain the following differential inequality after dropping the positive term $\nu|\nabla A_2 u|^2$

$$\frac{d}{dt}\|u\|^2 + \epsilon|Au|^2 \le c_1'|u|^2\|u\|^4 + \frac{2|f|^2}{\epsilon}. \tag{4.7}$$

We showed earlier that any solution will enter an absorbing ball B_{r_0} in H for $t \ge t_0(B_R)$. Thus

$$\frac{d}{dt}\|u\|^2 \le c_1'(r_0^2)\|u\|^4 + \frac{2|f|^2}{\epsilon}.$$

This result together with the estimate (4.5) allows us to apply the uniform Gronwall inequality with $y = \|u\|^2, g = c_1'(r_0^2)\|u\|^2$ and $h = \dfrac{2|f|^2}{\epsilon}$. we get

$$\|u(t)\|^2 \le \left(\sigma_0 + \frac{2|f|^2}{\epsilon}\right) \exp(c_1' r_0^2 \sigma_0), \quad \text{for } t \ge t_0(B_R) + 1,$$

where $t_0(B_R)$ is given in (4.4). This means that there exists an **absorbing ball** B_{r_1} of radius r_1 in V such that all the solutions will enter this ball after certain time:

$$W(t, 0; B_R) \subset B_{r_1}, \quad \text{for } t \ge t_0(B_R) + 1.$$

Since B_{r_1} is compact in H we conclude that $W(t, 0; \cdot)$ maps bounded sets in H into compact sets. Note that it follows from (4.7) that for $t \ge t_0(B_R) + 1$,

$$\int_t^{t+1} |Au|^2 d\tau \le \frac{1}{\epsilon}\left[c_1' r_0^2 r_1^4 + \frac{2|f|^2}{\epsilon} \right] = \sigma_1. \tag{4.8}$$

Now, let us set $w = A^2 u$ in (4.1) to get

$$\frac{1}{2}\frac{d}{dt}|Au|^2 + \epsilon\|Au\|^2 \le \nu|Au| \|Au\| + c_2|u| \|u\| \|Au\| + \|f\| \|Au\|.$$

Since any solution will be absorbed by balls B_{r_0} and B_{r_1} after time $t \ge t_0(B_R) + 1$, this becomes

$$\frac{d}{dt}|Au|^2 \le \frac{12\nu^2}{\epsilon}|Au|^2 + \frac{12c_2^2}{\epsilon}(r_0^2)(r_1^2) + \frac{12}{\epsilon}\|f\|^2.$$

Hence, we can apply the uniform Gronwall inequality again to conclude that

$$|A\boldsymbol{u}(t)|^2 \le \left[\, \sigma_1 + \frac{12}{\epsilon}(c_2' r_0^2 r_1^2 + \|f\|^2)\,\right] \exp(\frac{12\nu^2}{\epsilon}), \quad \text{for } t \ge t_0(B_R) + 2.$$

That is , there exists an **absorbing** ball B_{r_2} of radius r_2 in $D(A)$ such that all the solutions will enter this ball after time $t \ge t_0(B_R) + 2$:

$$W(t, 0; B_R) \subset B_{r_2}, \quad \text{for } t \ge t_0(B_R) + 2.$$

Note that

$$\text{closure of} \bigcup_{t \ge t_0(B_R)+1} W(t, 0; B_R)$$

is compact in H. Now, the compactness of the operators $W(t, 0; \cdot)$ in H implies that there exists a compact attractor Λ which attracts every bounded sets in H. In fact, Λ is the **global attractor** for the operators $W(t, 0; \cdot)$ and it is also the ω-limit set of absorbing set B_{r_2}, i.e. $\Lambda = \omega(B_{r_2})$. This means if we denote $W(t, 0; B_{r_2}) = B_{r_2}(t)$ then

$$\Lambda = \bigcap_{\tau \ge 0} cl\left(\bigcup_{t \ge \tau} B_{r_2}(\tau)\right).$$

Note that the global attractor Λ must be contained in the absorbing balls in H, V and $D(A)$:

$$\Lambda \subseteq B_{r_0} \cap B_{r_1} \cap B_{r_2}.$$

In addition one can show that if $W(t, 0; \cdot)$ is injective then in Λ, $W(t, 0; \cdot)$ will be defined for all $t \in \boldsymbol{R}$ [19]. Such a result for 2-D conventional Navier-Stokes equations has been proven by Ladyzhenskaya [10]. In the appendix we prove the time analyticity of the map $W(t, 0; \cdot)$ which implies injectivity.

Remark: Note that if we assume $f \in H$, then from the energy estimate we also get

$$\frac{d}{dt}|\boldsymbol{u}|^2 + (2\epsilon\mu_1 + \nu\eta_1)|\boldsymbol{u}|^2 \le \frac{|f|^2}{\nu\eta_1}.$$

By setting $\alpha' = 2\epsilon\mu_1 + \nu\eta_1$ and $\rho_0'^2 = \dfrac{|f|^2}{\nu\eta_1\alpha'}$ we obtain

$$|\boldsymbol{u}(t)|^2 \le |\boldsymbol{u}(0)|^2 e^{-\alpha't} + \rho_0'^2\,[\,1 - e^{-\alpha't}\,], \quad \alpha' > 0. \tag{4.9}$$

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Notice that the above estimate is uniform in ϵ. Let us denote $W(t,0;B_{r_0'}) = B_{r_0'}(t)$ and note that (4.9) implies that the absorbing ball $B_{r_0'}(t)$ will shrink as t increase. That is

$$B_{r_0'}(t_2) \subset B_{r_0'}(t_1), \quad \text{for } t_2 > t_1.$$

Note that from energy estimate we can also deduce,

$$\int_t^{t+1} |\nabla u|^2 d\tau \le \frac{1}{\nu}[\, r_0'^2 + \frac{1}{\nu\eta_1}|f|^2\,] = \sigma_0' \quad \text{for } t \ge t_0(B_{R'}).$$

Let us now set $w = A_1 u$ in (4.1),

$$\frac{1}{2}\frac{d}{dt}|\nabla u|^2 + \epsilon|\nabla A_2 u|^2 + \nu|A_1 u|^2 \le c_1|u|^{1/2}|\nabla u|\,|A_1 u|^{3/2} + |f|\,|A_1 u|. \qquad (4.10)$$

In the above we have used the following estimate for trilinear form $b(u,u,A_1u)$ which is valid only for $n = 2$:

$$|b(u,u,A_1u) \le c_1|u|^{1/2}|\nabla u||A_1 u|^{3/2}.$$

Using the Young's inequality in (4.10) and then dropping positive terms $\epsilon|\nabla A_2 u|^2$ and $\nu|A_1 u|^2$, we get

$$\frac{d}{dt}|\nabla u|^2 \le c_1'|u|^2|\nabla u|^4 + \frac{2|f|^2}{\nu}.$$

By applying the uniform Gronwall inequality again to obtain

$$|\nabla u(t)|^2 \le (\sigma_0' + \frac{2|f|^2}{\nu})\,\exp(c_1'\, r_0'^2\, \sigma_0'), \quad \text{for } t \ge t_0(B_{R'}) + 1.$$

Note that the estimate is again uniform in ϵ. This means that there exists an absorbing ball $B_{r_1'}$ in $V^* = \{u \in H_0^1(\Omega);\ \text{div} u = 0\}$ such that

$$W(t,0;B_{R'}) \subset B_{r_1'}, \quad \text{for } t \ge t_0(B_{R'}) + 1.$$

Let us denote the global attractor Λ_ϵ as ω-limit set of absorbing set $B_{r_1'}$. This means if we denote $W(t,0;B_{r_1'}) = B_{r_1'}(t)$ then

$$\Lambda_\epsilon = \bigcap_{\tau \ge 0} cl\left(\bigcup_{t \ge \tau} B_{r_1'}(\tau)\right).$$

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Note that since $W(t, 0; \cdot)$ maps a bounded set in H into compact set in H, Λ_ϵ is compact in H. (This is due to the compactness of embedding V^* in H). All the estimates used above are uniform in ϵ. Such results will be useful in establishing the limit of Λ_ϵ to the attractor Λ^* of the conventional Navier-Stokes equations as $\epsilon \to 0$. Special cases of such limit solutions were established in [15].

4.2 Formulation of Inertial Varieties

Let us now consider the system (2.1)-(2.4) as an equation of evolution in Hilbert space V:

$$\begin{cases} \dfrac{du}{dt} + \epsilon Au + \nu A_1 u + B(u, u) = f, & t > 0, \\[2mm] u(0) = u_0 \in V. \end{cases} \tag{4.11}$$

In the sequel, we will assume $f \in H$.

In the previous section we observed that although absorbing balls exists in H, V and $D(A)$ spaces, the positive invariance property can be established only for the H-ball B_{r_0}. However, we are interested only in the dynamics in a neighborhood of the attractor. We will thus devise a method to restrict our investigation to the dynamics inside the absorbing ball B_{r_1}. We will use a smooth cut-off funtion $\theta : R^+ \to [0, 1]$ and $\theta_{r_1}(r) = \theta(r/r_1)$ such that

$$\begin{cases} \theta(\xi) = 1, & \text{for } 0 \leq \xi \leq 1 \\[2mm] \theta(\xi) = 0, & \text{for } \xi \geq 2 \\[2mm] |\theta'(\xi)| \leq 2, & \text{for } \xi \geq 0, \end{cases}$$

to modify the equation (4.11):

$$\begin{cases} \dfrac{du}{dt} + \epsilon Au + \theta_{r_1}(\|u\|) R(u) = 0, \\[2mm] \text{where } R(u) = \nu A_1 u + B(u, u) - f. \end{cases} \tag{4.12}$$

The absorbing property of this modified equation can be shown by taking inner product with Au to (4.12). Let us consider the solution starting outside the ball B_{2r_1} in V, i.e.

$\|u\| \geq 2r_1$. Then we obtain the following simple differential inequality since $\theta_{r_1}(\|u\|) = 0$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + (\epsilon\mu_1 + \nu\eta_1)\|u\|^2 \leq \frac{1}{2}\frac{d}{dt}\|u\|^2 + \epsilon|Au|^2 + \nu|\nabla A_2 u|^2 = 0.$$

Here we have used the fact that $\mu_1\|u\|^2 \leq |Au|^2$ and $\eta_1\|u\|^2 \leq |\nabla A_2 u|^2$. This leads to

$$\|u(t)\|^2 \leq \|u(0)\|^2 e^{-\beta t}, \quad \text{with } \beta = 2(\epsilon\mu_1 + \nu\eta_1).$$

Notice that if $\|u(0)\| > r_3$ with $r_3 \geq 2r_1$, then the solution orbit will converge exponentially to a ball B_{r_3} of radius r_3. If $\|u(0)\| \leq r_3$, then solution orbit will stay inside the ball B_{r_3} and never leave this ball. That is, there exists an **exponentially absorbing and invariant ball** B_{r_3} in V for this modified equation (4.12). In addition, from the definition of the cut-off funtion θ, we have $\theta_{r_1}(\|u\|) = 1$ for $\|u\| \leq r_1$ (i.e. $u \in B_{r_1}$). This means that the original equation (4.11) and the modified equation (4.12) are identical in the neighborhood of the global attractor. Thus the the dynamics of (4.11) are exactly represented by system (4.12) after a certain time.

We shall define the solution operator $u_0 \to u(t)$ associated to the modified equation (4.12) as $W_m(t, 0; \cdot)$. The operator $W_m(t, 0; \cdot)$ is defined for all $t \geq 0$ and all $u_0 \in H$. The existence of the absorbing invariant ball $B_{r_3} \subset V$ implies that $W_m(t, 0; u_0) \in B_{r_3}$ for $t \geq T_c(u_0)$ for any initial condition $u_0 \in V$. We will thus construct the inertial variety $\mathcal{M}$ with a compact support in B_{r_3}. According to Foias, Sell and Temam [6], the formal definition of inertial variety is as follows: A set $\mathcal{M} \subset V$ is called an **inertial variety** of (4.12) if

1. $\mathcal{M}$ is a finite dimensional Lipschitz manifold;

2. $\mathcal{M}$ has a compact support and is invariant under the solution operator $W_m(t, 0; \cdot)$, i.e., $W_m(t, 0; \mathcal{M}) \subseteq \mathcal{M}$ for all $t \geq 0$;

3. all orbits of the solution of equation (4.12) in V are attracted exponentially to $\mathcal{M}$.

Remarks:

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(a) We begin with this formal definition given in [6] and then study the smoothness of such manifolds in section 4.5.

(b) Note that the condition 3 implies that $\Lambda \subset \mathcal{M}$. In fact, if $u_\Lambda \in \Lambda$, then since Λ is its own ω-limit set, for each $t > 0$, $\exists v_\Lambda \in \Lambda$ such that $u_\Lambda = W_m(t, 0; v_\Lambda)$. Now

$$\text{dist}(u_\Lambda, \mathcal{M}) = \text{dist}(W_m(t, 0; v_\Lambda), \mathcal{M}))$$

$$\leq e^{-\alpha t} \text{dist}(v_\Lambda, \mathcal{M}),$$

where α is a positive constant. Taking $t \to \infty$, we conclude that $\text{dist}(u_\Lambda, \mathcal{M})$ has to be zero. Since $\mathcal{M} \cap B_{r_1}$ is closed, we conclude that $u_\Lambda \in \mathcal{M}$.

Let us now define $p = P_N u$ and $q = Q_N u$ for $u \in V$. By applying the projection P_N and Q_N to the equation (4.12) we obtain evolution equations for $P_N V$ and $Q_N V$ respectively.

$$\frac{dp}{dt} + \epsilon A p + P_N F(u) = 0, \tag{4.13}$$

$$\frac{dq}{dt} + \epsilon A q + Q_N F(u) = 0, \tag{4.14}$$

where

$$F(u) = \theta_{r_1}(\|u\|) R(u).$$

Let us construct $\mathcal{M}$ as the graph of the Lipschitz function $\Phi : P_N V \to Q_N V$ with Φ belonging to the class $H_{b,l}$ defined below.

Definition 4.1 *Let $H_{b,l}$ be the space of Lipschitz maps Φ from $P_N V$ into $Q_N V$ which satisfy*

(i) $\|\Phi(p)\| \leq b, \quad b > 0, \quad \forall\, p \in P_N V,$

(ii) $\|\Phi(p_1) - \Phi(p_2)\| \leq l\, \|p_1 - p_2\|, \quad \forall\, p_1, p_2 \in P_N V,$

(iii) $\text{supp } \Phi \subseteq \{p \in P_N V : \|p\| \leq r_3\}.$

Note that $H_{b,l}$ is complete with the metric

$$\text{dist }(\Phi, \Phi') = \|\Phi - \Phi'\|_{H_{b,l}} = \sup_{p \in P_N V} \|\Phi(p) - \Phi'(p)\|.$$

Let $\Phi \in H_{b,l}$ and $p_0 \in P_N V$, then $p(t)$ will be determined by the following initial value problem:

$$\begin{cases} \dfrac{dp}{dt} + \epsilon Ap + P_N F(p + \Phi(p)) = 0 \\[2mm] p = p(t) \equiv p(t; \Phi, p_0) \\[2mm] p(0) = p_0 \end{cases} \qquad (4.15)$$

Let us first note that the modified nonlinear term $F(u)$ is globally bounded and Lipschitz.

Lemma 4.2 *For $\Phi \in H_{b,l}$ and $p_1, p_2 \in P_N V$, we have*

$$\text{(i)} \quad |F(u)| \le d_1, \qquad (4.16)$$

$$\text{(ii)} \quad |F(u_1) - F(u_2)| \le d_2 (1 + l) \|p_1 - p_2\|, \qquad (4.17)$$

where $d_2 = 2r_1^{-1} d_1 + \nu + 4c_0 r_1$ and $d_1 = 2\nu r_1 + 4c_0 r_1^2 + |f|$.

Proof: (i): Note that we have the following estimates for A_1 and $B(u, v)$

$$|A_1 u| \le \|u\|, \qquad \forall\, u \in V. \qquad (4.18)$$

$$|B(u, v)| \le c_0 \|u\|\,\|v\|, \qquad \forall\, u, v \in V. \qquad (4.19)$$

This gives

$$\begin{aligned} |F(u)| &\le \theta_{r_1}(\|u\|)\left[\nu|A_1 u| + |B(u,u)| + |f|\right] \\ &\le \theta_{r_1}(\|u\|)\left[\nu\|u\| + c_0\|u\|^2 + |f|\right] \\ &\le 2\nu r_1 + 4c_0 r_1^2 + |f| = d_1, \end{aligned} \qquad (4.20)$$

where d_1 is independent of time.

(ii) We write,

$$R(u_1) - R(u_2) = \nu A_1(u_1 - u_2) + B(u_1, u_1 - u_2) + B(u_1 - u_2, u_2).$$

Using (4.18) and (4.19) we get

$$|R(u_1) - R(u_2)| \le \nu\|u_1 - u_2\| + c_0\|u_1\|\,\|u_1 - u_2\| + c_0\|u_1 - u_2\|\,\|u_2\|.$$

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That is,

$$|R(\boldsymbol{u}_1) - R(\boldsymbol{u}_2)| \le (\nu + c_0\|\boldsymbol{u}_1\| + c_0\|\boldsymbol{u}_2\|)\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|. \tag{4.21}$$

Let us investigate

$$F(\boldsymbol{u}_1) - F(\boldsymbol{u}_2) = \theta_{r_1}(\|\boldsymbol{u}_1\|)R(\boldsymbol{u}_1) - \theta_{r_1}(\|\boldsymbol{u}_2\|)R(\boldsymbol{u}_2)$$

for the following three cases.

1. $\|\boldsymbol{u}_1\|, \|\boldsymbol{u}_2\| \ge 2r_1$,

2. $\|\boldsymbol{u}_1\| < 2r_1 \le \|\boldsymbol{u}_2\|$ or $\|\boldsymbol{u}_2\| < 2r_1 \le \|\boldsymbol{u}_1\|$,

3. $\|\boldsymbol{u}_1\|, \|\boldsymbol{u}_2\| < 2r_1$.

In the first case, since both $\theta_{r_1}(\|\boldsymbol{u}_1\|) = 0$ and $\theta_{r_1}(\|\boldsymbol{u}_2\|) = 0$ we have $F(\boldsymbol{u}_1) - F(\boldsymbol{u}_2) = 0$. Now if $\|\boldsymbol{u}_1\| < 2r_1 \le \|\boldsymbol{u}_2\|$ then since $\theta_{r_1}(\|\boldsymbol{u}_2\|) = 0$ we have $F(\boldsymbol{u}_1) - F(\boldsymbol{u}_2) = \theta_{r_1}(\|\boldsymbol{u}_1\|)R(\boldsymbol{u}_1)$. Thus

$$
\begin{aligned}
|F(\boldsymbol{u}_1) - F(\boldsymbol{u}_2)| \;&\le\; |\theta_{r_1}(\|\boldsymbol{u}_1\|) - \theta_{r_1}(\|\boldsymbol{u}_2\|)| \, |R(\boldsymbol{u}_1)| \\
&\le\; 2r_1^{-1}| \, \|\boldsymbol{u}_1\| - \|\boldsymbol{u}_2\| \, | \, |R(\boldsymbol{u}_1)| \\
&\le\; 2r_1^{-1}d_1\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|, \tag{4.22}
\end{aligned}
$$

where $d_1 = 2\nu r_1 + 4c_0 r_1^2 + |f|$. A similar result can be obtained for $\|\boldsymbol{u}_2\| < 2r_1 \le \|\boldsymbol{u}_1\|$. For the last case we have

$$F(\boldsymbol{u}_1) - F(\boldsymbol{u}_2) = [\,\theta_{r_1}(\|\boldsymbol{u}_1\|) - \theta_{r_1}(\|\boldsymbol{u}_2\|)\,]R(\boldsymbol{u}_1) + \theta_{r_1}(\|\boldsymbol{u}_2\|)[\,R(\boldsymbol{u}_1) - R(\boldsymbol{u}_2)\,].$$

From (4.21) and (4.22), we get

$$
\begin{aligned}
|F(\boldsymbol{u}_1) - F(\boldsymbol{u}_2)| \;&\le\; \Big\{\, 2r_1^{-1}d_1 + \nu + 4c_0 r_1 \,\Big\} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\| \\
&\le\; d_2\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|, \tag{4.23}
\end{aligned}
$$

with $d_2 = 2r_1^{-1}d_1 + \nu + 4c_0 r_1$.

Let us estimate

$$u_1 - u_2 = p_1 - p_2 + [\Phi(p_1) - \Phi(p_2)].$$

as,

$$\|u_1 - u_2\| \le (1 + l)\|p_1 - p_2\|.$$

Combining this with (4.23) gives

$$|F(u_1) - F(u_2)| \le d_2 (1 + l)\|p_1 - p_2\|.$$

$\square$

In consequence of (4.17), we have

$$|P_N F(p_1 + \Phi(p_1)) - P_N F(p_2 + \Phi(p_2))| \le d_2 (1 + l)\|p_1 - p_2\|, \quad p_1, p_2 \in P_N V.$$

This means $P_N F$ is a mapping from $P_N V$ into $P_N H$ and satisfies global Lipschitz condition. From a uniqueness theorem for evolution equations in finite dimensions [2,3], there exists at most one solution $p(t) \equiv p(t; \Phi, p_0)$ for the Cauchy problem (4.15).

Let us now integrate (4.14) from ξ to t to get

$$q(t) = Q_N S(t - \xi)q(\xi) + \int_\xi^t Q_N S(t - \tau)Q_N F(u(\tau))d\tau.$$

Notice that (4.16) implies that $Q_N F(p + \Phi(p))$ is a bounded operator such that $Q_N F : P_N V \to Q_N H$. Hence, there exists a unique solution $q(t) \equiv q(t; \Phi, p_0)$ that stays bounded as $\xi \to -\infty$. This can be seen as follows: from Lemma 3.3

$$\|Q_N S(t - \xi)q(\xi)\| \le \mu_{N+1}^{1/2} e^{\epsilon \mu_{N+1}(\xi - t)} \quad \to 0 \text{ as } \xi \to -\infty.$$

Also it follows from (4.16) that

$$\int_{-\infty}^t \|Q_N S(t - \tau)Q_N F(u(\tau))\|d\tau$$
$$\le \int_{-\infty}^t \|Q_N S(t - \tau)\|_{\mathcal{L}(Q_N H; Q_N V)} |Q_N F(u(\tau))|d\tau$$
$$\le d_1 \int_{-\infty}^t \|Q_N S(t - \tau)\|_{\mathcal{L}(Q_N H; Q_N V)}d\tau$$
$$\le d_1 \epsilon^{-1}\mu_{N+1}^{-1/2} \quad \text{for } t \in R. \tag{4.24}$$

We now take $\xi = -\infty$

$$q(t) = -\int_{-\infty}^{t} Q_N S(t - \tau) Q_N F(u(\tau)) d\tau. \qquad (4.25)$$

Here $S(\xi)$ is the holomorphic semigroup generated by $-\epsilon A$ described in section 3.1. Since $q(t)$ is continuous and bounded for all $t \in \mathbf{R}$, we choose $t = 0$ so that $p_0 \in P_N V$ will be related to $q(0) \equiv q(0; \Phi, p_0)$ by

$$q(0) = -\int_{-\infty}^{0} Q_N S(-\tau) Q_N F(u(\tau)) d\tau.$$

We thus seek the (unique) uniformly bounded solution of (4.11) on $(-\infty, 0]$ with $u = p + \Phi(p)$. We will now define a mapping $T : p_0 \to q(0)$ as

$$[T\Phi](p_0) := -\int_{-\infty}^{0} Q_N S(-\tau) Q_N F(u(\tau)) d\tau, \quad \check{\Phi} \in H_{b,l}, \qquad (4.26)$$

where $u(\tau) = p(\tau; \Phi, p_0) + \Phi(p(\tau; \Phi, p_0))$. Notice that the condition that $u = p_0 + q(0)$ belonging to $\mathcal{M}$ is equivalent to the existence of a fixed point for the map T:

$$q(0) = \Phi(p_0) = T\Phi(p_0), \quad \forall\, p_0 \in P_N V.$$

4.3   Inertial Manifold Theorem

We will establish the existence of inertial manifold using contraction mapping theorem. For this purpose we must prove the following:

Lemma 4.3 *For* $\Phi \in H_{b,l}$, *if* $\mu_{N+1}^{1/2} - \mu_N^{1/2} \geq \epsilon^{-1} d_3$ *and* $l \leq 2$ *then the* T *maps* $H_{b,l}$ *into itself:*

$$T : H_{b,l} \longrightarrow H_{b,l},$$

where $d_3 = 2d_2(1 + l)l^{-1}$ *and* l *is the Lipschitz constant for* Φ.

Proof: We first note that,

Lemma 4.4 *For* $\Phi \in H_{b,l}$, *we have*

$$\operatorname{supp} T\Phi \subseteq \{p \in P_N V : \|p\| \leq r_3\}, \quad \text{where } r_3 \geq 2r_1.$$

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Proof: We need to show for $\|p_0\| \geq r_3$ we have $T\Phi(p_0) = 0$ for all $\Phi \in H_{b,l}$. Notice that $u = P_N p + Q_N \Phi(p)$ since $P_N^2 = P_N$ and $Q_N^2 = Q_N$. Now $A^{1/2} \dot{u} = P_N A^{1/2} p + Q_N A^{1/2} \Phi(p)$ since P_N, Q_N commute with $A^{1/2}$. Thus, $\|u\|^2 = \|p\|^2 + \|\Phi(p)\|^2$. Hence for $\|p\| > 2r_1$ we have $\|u\| \geq \|p\| > 2r_1$. This implies $\theta_{r_1}(\|u\|) = 0$ for $\|p\| > 2r_1$.

Let us consider the initial point outside the ball B_{r_3} such that $\|p_0\| > r_3 \geq 2r_1$. We can then find a time t such that $\|p(t)\| > 2r_1$, for $\tau \leq t \leq 0$ and hence $\theta_{r_1}(\|u\|) = 0$. Thus (4.15) becomes,

$$
\begin{cases}
\dfrac{dp}{dt} + \epsilon A p = 0 \\[2mm]
p(0) = p_0.
\end{cases}
$$

By taking inner product with Ap for above system and using the fact $|Ap|^2 \geq \mu_1 \|p\|^2$ we get

$$
\frac{d}{dt} \|p\|^2 + 2\epsilon \mu_1 \|p\|^2 \leq 0.
$$

This gives us

$$
\|p(0)\| \leq \|p(\tau)\| e^{\epsilon \mu_1 \tau} \leq \|p(\tau)\|, \quad \tau < 0.
$$

Since $\|p_0\| > r_3 \geq 2r_1$, we have $\|p(\tau)\| > r_3 \geq 2r_1$. This means that $u(\tau)$ always stays outside the ball B_{r_3} for all $\tau < 0$. Therefore, $\theta_{r_1}(\|u\|) = 0$ for all $\tau < 0$. This gives us $F(u) = 0$ and thus $T\Phi(p_0) = 0$.

$\square$

Let us show that $T\Phi$ is globally bounded in V.

Lemma 4.5 *If $p_0 \in P_N V$, then $[T\Phi](p_0) \in Q_N V$ and*

$$
\|T\Phi(p_0)\| \leq b,
$$

with $b = 2d_1 \epsilon^{-1} e^{-1/2} \mu_{N+1}^{-1/2}$.

Proof: From the definition of the map T it is obvious that $T\Phi(p_0) \in Q_N V$ and

$$
\|T\Phi(p_0)\| \leq \int_{-\infty}^0 \|Q_N S(-\tau)\|_{\mathcal{L}(Q_N H; Q_N V)} |Q_N F(u)| d\tau.
$$

Recalling the estimate for $\|Q_N S(-\tau)\|_{\mathcal{L}(Q_N H; Q_N V)}$ from Lemma 3.3 (with $\alpha = 1/2$) we get

$$\int_{-\infty}^{0} \|Q_N S(-\tau)\|_{\mathcal{L}(Q_N H; Q_N V)} d\tau \leq 2\epsilon^{-1} e^{-1/2} \mu_{N+1}^{-1/2}. \qquad (4.27)$$

It follows from (4.27) and (4.16) that

$$\|T\Phi(p_0)\| \leq 2d_1 \epsilon^{-1} e^{-1/2} \mu_{N+1}^{-1/2} = b. \qquad (4.28)$$

$\square$

Let us now prove the global Lipschitz continuity of the map $[T\Phi](\cdot)$.

Lemma 4.6 *If*

$$\delta_N = \epsilon(\mu_{N+1} - \mu_N) - d_2(1 + l)\mu_N^{1/2} > 0,$$

then for $\Phi \in H_{b,l}$ and for $p_{01}, p_{02} \in P_N V$ we have

$$\| [T\Phi](p_{01}) - [T\Phi](p_{02}) \| \leq l' \|p_{01} - p_{02}\|, \qquad (4.29)$$

$$with \quad l' = d_2(1 + l)\mu_{N+1}^{-1/2} \left[\epsilon^{-1} + (\epsilon - \sigma_N \xi_N)^{-1} \right] e^{-1/2} \exp \frac{\sigma_N \xi_N}{2\epsilon} \qquad (4.30)$$

$$and \quad \sigma_N = \frac{\mu_N}{\mu_{N+1}}, \quad \xi_N = \epsilon + d_2(1 + l)\mu_N^{-1/2}.$$

Proof: Let Φ be a fixed element in $H_{b,l}$. Let $p_1 = p_1(t)$ and $p_2 = p_2(t)$ be two different solutions of the initial value problem (4.15) with $p_1(0) = p_{01}, p_2(0) = p_{02}$ respectively.

$$\begin{cases} \dfrac{dp_1}{dt} + \epsilon A p_1 + P_N F(u_1) = 0 \\[2mm] p_1(0) = p_{01} \end{cases} \qquad (4.31)$$

and

$$\begin{cases} \dfrac{dp_2}{dt} + \epsilon A p_2 + P_N F(u_2) = 0 \\[2mm] p_2(0) = p_{02}, \end{cases} \qquad (4.32)$$

where $u_1 = p_1 + \Phi(p_1)$ and $u_2 = p_2 + \Phi(p_2)$. By subtracting (4.32) from (4.31), we get

$$\frac{dp}{dt} + \epsilon A p + P_N F(u_1) - P_N F(u_2) = 0, \qquad (4.33)$$

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where $p(t) = p_1(t) - p_2(t)$ and $p(0) = p_{01} - p_{02}$.

We take inner product of Ap with (4.33) and use estimate (4.17) to get

$$\frac{1}{2}\frac{d}{dt}\|p\|^2 + \epsilon|Ap|^2 \leq |P_N[F(u_1) - F(u_2)]|\,|Ap|$$

$$\leq d_2(1 + l)\|p\|\,|Ap|.$$

By using the fact $|Ap|^2 \leq \mu_N\|p\|^2$ we obtain

$$\|p\|\frac{d}{dt}\|p\| \geq -\left[\epsilon\mu_N + d_2(1 + l)\mu_N^{1/2}\right]\|p\|^2,$$

which implies

$$\frac{d}{dt}\|p\| + \mu_N\xi_N\|p\| \geq 0, \quad \text{where } \xi_N = \epsilon + d_2(1 + l)\mu_N^{-1/2}.$$

This leads to

$$\|p_1(\tau) - p_2(\tau)\| \leq \|p_{01} - p_{02}\|\exp\{-\mu_N\xi_N\tau\}, \quad \forall\,\tau \leq 0. \tag{4.34}$$

Now, using the Lipschitz condition on $F(u)$ given in (4.17), we can estimate

$$\|T\Phi(p_{01}) - T\Phi(p_{02})\| \leq \int_{-\infty}^{0} \|Q_N S(-\tau)\|_{\mathcal{L}(Q_N H;Q_N V)}|Q_N[F(u_1) - F(u_2)]|d\tau$$

$$\leq d_2(1 + l)\int_{-\infty}^{0} \|Q_N S(-\tau)\|_{\mathcal{L}(Q_N H;Q_N V)}\|p_1(\tau) - p_2(\tau)\|d\tau$$

By applying result (4.34) we get

$$\|T\Phi(p_{01}) - T\Phi(p_{02})\| \leq d_2(1 + l)\|p_{01} - p_{02}\|\{\mu_{N+1}^{1/2}\int_{-\infty}^{-1/2\epsilon\mu_{N+1}}\exp[\,\delta_N\tau\,]d\tau$$

$$+ \left(\frac{1}{2\epsilon e}\right)^{1/2}\int_{-1/2\epsilon\mu_{N+1}}^{0}|\tau|^{-1/2}\exp[\,-\mu_N\xi_N\tau\,]d\tau\},$$

where $\delta_N = \epsilon(\mu_{N+1} - \mu_N) - d_2(1 + l)\mu_N^{1/2}$. Note that

$$\mu_{N+1}^{1/2}\int_{-\infty}^{-1/2\epsilon\mu_{N+1}}e^{\delta_N\tau}d\tau \leq e^{-1/2}\mu_{N+1}^{-1/2}(\epsilon - \sigma_N\xi_N)^{-1}e^{\frac{\sigma_N\xi_N}{2\epsilon}}, \tag{4.35}$$

with $\delta_N > 0$ and $\sigma_N = \mu_N/\mu_{N+1}$. Also

$$\left(\frac{1}{2\epsilon e}\right)^{1/2}\int_{-1/2\epsilon\mu_{N+1}}^{0}|\tau|^{-1/2}e^{-\mu_N\xi_N\tau}d\tau \leq \epsilon^{-1}e^{-1/2}\mu_{N+1}^{-1/2}e^{\frac{\sigma_N\xi_N}{2\epsilon}}. \tag{4.36}$$

We thus get,

$$\|T\Phi(\boldsymbol{p}_{01}) - T\Phi(\boldsymbol{p}_{02})\| \leq l' \, \|\boldsymbol{p}_{01} - \boldsymbol{p}_{02}\|,$$

with

$$l' = d_2(1+l)\mu_{N+1}^{-1/2} \left[\epsilon^{-1} + (\epsilon - \sigma_N \xi_N)^{-1} \right] e^{-1/2} \exp \frac{\sigma_N \xi_N}{2\epsilon}.$$

$\square$

Now in order for the transform T to map $H_{b,l}$ into itself, we must have $l' \leq l$. An elementary calculation shows that sufficient conditions for $l' \leq l$ are

$$\epsilon^{-1} d_3 \leq \mu_{N+1}^{1/2}, \qquad \text{with } d_3 = 2d_2(1+l)l^{-1} \quad \text{and}$$

$$\epsilon \sigma_N^{1/2} + d_3 \, \mu_{N+1}^{-1/2} \leq \epsilon, \qquad \text{with } l \leq 2.$$

Combining above two results,

$$\mu_{N+1}^{1/2} - \mu_N^{1/2} \geq \epsilon^{-1} d_3, \quad \text{with } d_3 = 2d_2(1+l)l^{-1} \text{ and } l \leq 2.$$

$\square$

Lemma 4.7 *For $\Phi \in H_{b,l}$, if $\mu_{N+1}^{1/2} \geq \epsilon^{-1} d_4$ and $l \leq 2$ then the transform T is a strict contraction on $H_{b,l}$. Here $d_4 = 2d_2(2e^{-1/2} + l)$.*

Proof: We will first show that for $\Phi_1, \Phi_2 \in H_{b,l}$ and $\boldsymbol{p}_0 \in P_N V$,

$$\| \left[T\Phi_1(\boldsymbol{p}_0) - T\Phi_2(\boldsymbol{p}_0) \right] \| \leq L \, \|\Phi_1 - \Phi_2\|.$$

Consider two elements Φ_1 and Φ_2 in $H_{b,l}$. We take $\boldsymbol{u}_1 = \boldsymbol{p}_1 + \Phi_1(\boldsymbol{p}_1)$ and $\boldsymbol{u}_2 = \boldsymbol{p}_2 + \Phi_2(\boldsymbol{p}_2)$ with same initial condition $\boldsymbol{p}_0$. Then analogous to (4.33) we get

$$\begin{cases} \dfrac{d\boldsymbol{p}}{dt} + \epsilon A\boldsymbol{p} + P_N F(\boldsymbol{u}_1) - P_N F(\boldsymbol{u}_2) = 0 \\[2mm] \boldsymbol{p}(0) = 0. \end{cases}$$

By taking inner product with $A\boldsymbol{p}$ and again using the fact $|A\boldsymbol{p}|^2 \leq \mu_N \|\boldsymbol{p}\|^2$ we get

$$\begin{cases} \dfrac{d}{dt}\|\boldsymbol{p}\| + \mu_N \xi_N \|\boldsymbol{p}\| \geq -d_2 \mu_N^{1/2} \|\Phi_1 - \Phi_2\| \\[2mm] \boldsymbol{p}(0) = 0, \end{cases}$$

where $\xi_N = \epsilon + d_2(1+l)\mu_N^{-1/2} > \epsilon$. This gives us

$$\|p_1(\tau) - p_2(\tau)\| \leq \frac{d_2}{\epsilon} \mu_N^{-1/2} \|\Phi_1 - \Phi_2\| \exp\{-\mu_N \xi_N \tau\}. \tag{4.37}$$

In addition, we have

$$\|u_1 - u_2\| \leq (1+l)\|p_1 - p_2\| + \|\Phi_1 - \Phi_2\|_{H_{b,l}}.$$

Thus combining with (4.23) we get

$$|F(u_1) - F(u_2)| \leq d_2\left[(1+l)\|p_1 - p_2\| + \|\Phi_1 - \Phi_2\|\right]. \tag{4.38}$$

This gives

$$\|T\Phi_1(p_0) - T\Phi_2(p_0)\| \leq$$
$$d_2 \int_{-\infty}^0 \|Q_N S(-\tau)\|_{\mathcal{L}(Q_N H; Q_N V)}\left[(1+l)\|p_1(\tau) - p_2(\tau)\| + \|\Phi_1 - \Phi_2\|\right]d\tau.$$

By applying the estimate (4.37) we get

$$\|T\Phi_1(p_0) - T\Phi_2(p_0)\| \leq$$
$$d_2\|\Phi_1 - \Phi_2\| \int_{-\infty}^0 \|Q_N S(-\tau)\|_{\mathcal{L}(Q_N H; Q_N V)}\left[1 + \frac{d_2}{\epsilon}\mu_N^{-1/2}(1+l)e^{-\mu_N \xi_N \tau}\right]d\tau.$$

From (4.27), (4.35) and (4.36), we finally get

$$\|T\Phi_1(p_0) - T\Phi_2(p_0)\| \leq \epsilon^{-1}d_2\left(2e^{-1/2}\mu_{N+1}^{-1/2} + \mu_N^{-1/2}\,l'\right)\|\Phi_1 - \Phi_2\|$$
$$= L\,\|\Phi_1 - \Phi_2\|,$$

where l' is given in (4.30).

In order for T to be a strict contraction we must have $L < 1$. Let us choose

$$L = d_2\epsilon^{-1}\left(2e^{-1/2}\mu_{N+1}^{-1/2} + \mu_N^{-1/2}\,l'\right) \leq \frac{1}{2}.$$

Since $l' \leq l$ and $\mu_N^{1/2} \leq \mu_{N+1}^{1/2}$, a sufficient condition for $L \leq 1/2$ is

$$\mu_{N+1}^{1/2} \geq \epsilon^{-1}d_4, \quad \text{with } d_4 = 2d_2\left(2e^{-1/2} + l\right) \text{ and } l \leq 2.$$

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$\square$

Let us now establish the existence of the inertial manifold for (4.12).

Theorem 4.1 *Let* $\Phi \in H_{b,l}$ *with* $0 < l < 1$ *and* b *be given in (4.28). Let* N^* *be given and for* $R = r_3 + b$, *along with* $0 < \gamma < 1 - l$, *there exist constants* d_3, d_4 *depending only on* l, r_1, ν, f *and* Ω *such that the following three conditions are satisfied:*

(i) $N \geq N^*$;

(ii) $\mu_{N+1}^{1/2} \geq \epsilon^{-1} d_4$;

(iii) $\mu_{N+1}^{1/2} - \mu_N^{1/2} \geq \epsilon^{-1} d_3$.

Then the transform T *has a fixed point* $\Phi \in H_{b,l}$. *The manifold* $\mathcal{M}$ *defined by the graph of* Φ *is an inertial manifold for (4.12).*

Proof: From Lemma 4.3 and Lemma 4.7, we see that the transform T maps $H_{b,l}$ into itself and is a strict contraction. Thus there exists a unique fixed point $\Phi^* \in H_{b,l}$ such that $T\Phi^* = \Phi^*$ by the contraction mapping theorem.

We note here that by construction the manifold $\mathcal{M}$ is invariant to the action of the nonlinear evolution operator $W_m(t, 0; \cdot)$. To see this we first write the relationship $T\Phi^* = \Phi^*$ as

$$\Phi^*(p_0) = -\int_{-\infty}^0 Q_N S(-\tau) Q_N F(u(\tau; \Phi^*, p_0)) d\tau, \tag{4.39}$$

with $u(\tau, p_0) = p(\tau; \Phi^*, p_0) + \Phi^*(p(\tau; \Phi^*, p_0))$.

Let $p(t) = p(t; \Phi^*, p_0)$ be the solution of (4.15) that is defined for all $t \in \mathbf{R}$. Then we need to show that $q(t) = \Phi^*(p(t))$ is a solution of (4.14). Let us consider $\Phi^*(p(t))$ and using the fact $u(\tau, p(t)) = u(\tau, W_m(t, 0; p_0)) = u(\tau + t; p_0)$ to get

$$\Phi^*(p(t)) = -\int_{-\infty}^0 Q_N S(-\tau) Q_N F(u(\tau, p(t))) d\tau$$

$$= -\int_{-\infty}^0 Q_N S(-\tau) Q_N F(u(\tau + t, p_0)) d\tau$$

By setting $\tau = \eta - t$ and from (4.24), (4.25) one has

$$\Phi^*(p(t)) = -\int_{-\infty}^{t} Q_N S(t-\eta) Q_N F(u(\eta, p_0)) d\eta, \quad \text{for all } t \in R.$$

By making use of Leibnitz rule formally, we find that

$$\frac{d\Phi^*(t)}{dt} + \epsilon A Q_N \left[-\int_{-\infty}^{t} Q_N S(t-\eta) Q_N F(u(\eta, p_0)) d\eta \right] + Q_N F(u(t, p_0)) = 0.$$

This gives

$$\frac{d\Phi^*(t)}{dt} + \epsilon A Q_N \Phi^*(p(t)) + Q_N F(p(t) + \Phi^*(p(t))) = 0$$

Clearly, we see that $(p(t), \Phi^*(p(t)))$ is a solution of (4.13), (4.14). Hence $u(t)$ is a solution of (4.12) with $q(t) = \Phi^*(p(t))$. This proves the invariance property $W_m(t, 0; \mathcal{M}) \subseteq \mathcal{M}$.

We shall now establish exponential attractive property of the manifold $\mathcal{M}$. Let us recall a theorem on the squeezing property of solution orbits [15].

Theorem 4.2 *Let $W_m(t, 0; u_0)$ and $W_m(t, 0; v_0)$ be two solution of (4.12) with $u_0, v_0 \in B_R \subset V$. Then there exist constants C_1, C_2 depending only on R, T, f, ϵ, ν, γ and Ω such that for every $\gamma > 0$ and every $t \in [0, T]$, we have either*

$$\|Q_N[(W_m(t, 0; u_0) - W_m(t, 0; v_0)]\| \leq \gamma \|P_N[W_m(t, 0; u_0) - W_m(t, 0; v_0)]\|$$

or

$$\|W_m(t, 0; u_0) - W_m(t, 0; v_0)\| \leq C_1 \exp(-C_2 \epsilon \mu_{N+1} t) \|u_0 - v_0\| \quad (4.40)$$

for every $N \geq 1$.

We will now show that this property implies that the manifold $\mathcal{M}$ be globally exponential attracting. For convenience we choose N^* such that

$$\mu_{N^*+1} \geq \frac{2C_2 \epsilon t_0}{\log(2C_1)}.$$

This gives

$$\|Q_N[W_m(t, 0; u_0) - W_m(t, 0; v_0)]\| \leq \gamma \|P_N[W_m(t, 0; u_0) - W_m(t, 0; v_0)]\| \quad (4.41)$$

and

$$\|W_m(t, 0; \boldsymbol{u}_0) - W_m(t, 0; \boldsymbol{v}_0)\| \leq \frac{1}{2} \|\boldsymbol{u}_0 - \boldsymbol{v}_0\| \qquad (4.42)$$

for $t_0 \leq t \leq 2t_0$.

Let us denote the distance between any point $\boldsymbol{w}$ in the absorbing ball B_{r_3} in V and the manifold $\mathcal{M}$ by

$$\operatorname{dist}(\boldsymbol{w}, \mathcal{M}) = \inf\{\|\boldsymbol{w} - \boldsymbol{v}\| : \boldsymbol{v} \in \mathcal{M}\}.$$

Let $\boldsymbol{v}_0 \in \mathcal{M}$ so that $\boldsymbol{v}_0 = P_N \boldsymbol{v}_0 + \Phi(P_N \boldsymbol{v}_0)$ and that $\|\boldsymbol{u}_0 - \boldsymbol{v}_0\| = \operatorname{dist}(\boldsymbol{u}_0, \mathcal{M})$. Moreover, it can be shown that $\|P_N \boldsymbol{v}_0\| \leq r_3$ for $\boldsymbol{v}_0 \in \mathcal{M}$. Hence for $\Phi \in H_{b,t}$, it follows at once that

$$\|\boldsymbol{v}_0\| \leq \|P_N \boldsymbol{v}_0\| + \|\Phi(P_N \boldsymbol{v}_0)\| \leq r_3 + b.$$

hence we can choose $R = r_3 + b$ such that $\boldsymbol{u}_0, \boldsymbol{v}_0 \in B_R$. Let us now apply the squeezing property stated above. We shall first establish the attracting property in $t_0 \leq \tau \leq 2t_0$ using either one of conditions (4.41),(4.42).

Using (4.42), we get

$$\operatorname{dist}(W_m(t, 0; \boldsymbol{u}_0), \mathcal{M}) \leq \|W_m(t, 0; \boldsymbol{u}_0) - W_m(t, 0; \boldsymbol{v}_0)\|$$

$$\leq \frac{1}{2}\|\boldsymbol{u}_0 - \boldsymbol{v}_0\| = \frac{1}{2}\operatorname{dist}(\boldsymbol{u}_0, \mathcal{M}).$$

Note that $\boldsymbol{v}_0$ will always stay on the manifold for $t_0 \leq \tau \leq 2t_0$.

Let us consider a point $\boldsymbol{u}(t)$ which has evolved from $\boldsymbol{u}_0 \in B_R$. That is $\boldsymbol{u}(t) = W_m(t, 0; \boldsymbol{u}_0)$. Then there is a point on the manifold $\mathcal{M}$ which can be represented as $\boldsymbol{v}^* = P_N W_m(t, 0; \boldsymbol{u}_0) + \Phi(P_N W_m(t, 0; \boldsymbol{u}_0))$. It follows from (4.41) that

$$\operatorname{dist}(W_m(t, 0; \boldsymbol{u}_0), \mathcal{M}) \leq \|W_m(t, 0; \boldsymbol{u}_0) - \boldsymbol{v}^*\|$$

$$\leq \|Q_N W_m(t, 0; \boldsymbol{u}_0) - \Phi(P_N W_m(t, 0; \boldsymbol{u}_0))\|$$

$$\leq \|Q_N W_m(t, 0; \boldsymbol{u}_0) - Q_N W_m(t, 0; \boldsymbol{v}_0)\|$$

$$+ \|\Phi(P_N W_m(t, 0; \boldsymbol{v}_0)) - \Phi(P_N W_m(t, 0; \boldsymbol{u}_0))\|.$$

Since $\mathcal{M}$ is invariant to the action of $W_m(t, 0; \cdot)$, we have $Q_N W_m(t, 0; v_0) = \Phi(P_N W_m(t, 0; v_0))$. Note that we also have

$$\|\Phi(P_N W_m(t, 0; v_0)) - \Phi(P_N W_m(t, 0; u_0))\|$$

$$\leq l \|P_N W_m(t, 0; v_0) - P_N W_m(t, 0; u_0)\|.$$

From this it follows that

$$\text{dist}(W_m(t, 0; u_0), \mathcal{M}) \leq (\gamma + l) \|P_N W_m(t, 0; v_0) - P_N W_m(t, 0; u_0)\|$$

$$\leq (\gamma + l) \|W_m(t, 0; v_0) - W_m(t, 0; u_0)\|$$

$$\leq \frac{1}{2} \|u_0 - v_0\| = \frac{1}{2} \text{dist}(u_0, \mathcal{M}).$$

Here note that $\mu_N \leq \mu_{N+1}$ and $\gamma + l \leq 1$.

From the semigroup property of $W_m(t, 0; \cdot)$, we can deduce that

$$\text{dist}(W_m(t, 0; u_0), \mathcal{M}) \leq \left(\frac{1}{2}\right)^n \text{dist}(u_0, \mathcal{M}).$$

Suppose we write $t = n\tau$ with $t_0 \leq \tau \leq 2t_0$, then

$$\text{dist}(W_m(t, 0; u_0), \mathcal{M}) \leq e^{-\frac{t}{2t_0}\ln 2}\text{dist}(u_0, \mathcal{M}).$$

This means that

$$\text{dist}(W_m(t, 0; u_0), \mathcal{M}) \longrightarrow 0 \quad \text{exponentially}, \quad \text{as } t \to \infty.$$

$\square$

4.4 Spectral Growth Rate

Let us now discuss the spectral gap condition for operator A. As established in the previous sections, a sufficient condition for the existence of inertial manifold is

$$\mu_{N+1}^{1/2} - \mu_N^{1/2} \geq \epsilon^{-1} d_3, \tag{4.43}$$

for each $\epsilon > 0$.

Let us consider the following two eigenvalue problems:

$$\begin{cases} (-\Delta)^2 \psi = \nu \psi \\ \\ \psi|_{\partial\Omega} = 0, \quad \Delta\psi|_{\partial\Omega} = 0 \end{cases} \tag{4.44}$$

and

$$\begin{cases} (-\Delta)^2 \phi + \nabla p = \mu \phi \\ \\ \nabla \cdot \phi = 0 \\ \\ \phi|_{\partial\Omega} = 0, \quad \Delta\phi|_{\partial\Omega} = 0 \end{cases} \tag{4.45}$$

The eigenvalues ν_N of (4.44) can be obtained by the minimization problem:

$$\nu_N = \inf_{u \in M_{N-1}^\perp} \frac{(\Delta u, \Delta u)}{(u, u)} = \inf_{u \in M_{N-1}^\perp} \frac{|\Delta u|^2}{|u|^2}, \quad u \in \hat{V},$$

where $\hat{V} = \{u \in H^2(\Omega), u|_{\partial\Omega} = 0\}$ and $M_{N-1} = \text{span}\{\psi_1, \cdots, \psi_{N-1}\}$. Similarly, for eigenvalues μ_N of (4.45), we minimize the Rayleigh quotient with u restricted to the space V. Since V is a subspace of $\hat{V}$, by First Monotonicity Principle [22] we can conclude

$$\mu_N \geq \nu_N.$$

It can be shown that the growth rate of the eigenvalues ν_N of $(-\Delta)^2$ for an arbitrary smooth bounded domain $\Omega \subset \mathbf{R}^n$ are [4,17]

$$\nu_N \sim 16\pi^4 \left(\frac{N}{B_N \mathcal{V}}\right)^{4/n} \quad \text{as } N \to \infty.$$

Here $\mathcal{V}$ is the volume of Ω and B_n is the volume of the unit ball in $\mathbf{R}^n$. Since $\mu_N \geq \nu_N$, we also have

$$\mu_N \sim c\, N^{4/n} \quad \text{as } N \to \infty.$$

Here μ_N is eigenvalues of dissipation operator $A = P_H(-\Delta)^2$ and n is the space dimension.

In particular, for space dimension $n = 2$, we have $\mu_N \sim c\, N^2$. It is easy to see that for sufficient large N we get

$$\mu_{N+1}^{1/2} - \mu_N^{1/2} \sim c^{1/2}.$$

This indicates that there exists an inertial manifold only for $\epsilon > \dfrac{d_3}{c^{1/2}}$ instead of each $\epsilon > 0$. Notice that if eigenvalues $\mu_N \sim c\, N^{2+\delta}$ as $N \to \infty$, then the large spectral gap condition

in (4.43) is satified for every $\epsilon > 0$. This will be true if $A = P_H(-\Delta)^{2+\delta}$, $\delta > 0$. Hence the dissipation operator A associated with regularized Navier-Stokes equations in arbitrary two dimensional domains, the spectral gap condition is marginal. However the spectral gap condition for A is satisfied for the following case.

Let us consider problem (4.45) with periodic boundary conditions. Using Fourier series we get the eigenvalues as

$$\mu_{\vec{k}} = \frac{16\pi^4}{L^4}|\vec{k}|^4, \quad \text{with } \vec{k} = (k_1, \cdots, k_n).$$

In particular, for $n = 2$ we have

$$\mu_{(k_1,k_2)} = \frac{16\pi^4}{L^4}(k_1^2 + k_2^2)^2.$$

Let us now set $S_{(k_1,k_2)} = k_1^2 + k_2^2$, thus $\mu_{(k_1,k_2)} = c\, S_{(k_1,k_2)}^2$ with $c = 16\pi^4/L^4$. According to magnitude of $S_{(k_1,k_2)}$, we can rewrite $S_{(k_1,k_2)}$ in a sequence as $S_1 \leq S_2 \leq \cdots$ and this will establish an ordering of the eigenvalues as $\mu_1 \leq \mu_2 \leq \cdots$, with $\mu_N = c\, S_N^2$. Since S_N is sum of two squares, we can apply a number theoretic result by Richards [18]:

$$\mu_{N+1}^{1/2} - \mu_N^{1/2} = c^{1/2}(S_{N+1} - S_N) \geq \frac{1}{4}\log S_N \quad \text{as } N \to \infty.$$

Here we note that when Ω is periodic, A has a zero eigenvalue. Hence, in order for the earlier theories to apply we should modify A by $A + \beta$, $\beta > 0$.

4.5 Regularity of Inertial Varieties

In this section we will study the regularity properties of inertial manifolds $\mathcal{M}$. We will obtain a sufficient condition for the inertial variety to be C^1. We will prove in particular that once the existence of inertial variety is established then higher dimensional manifolds are automatically C^1.

Definition 4.2 *Let $H_{b,l}^1$ be the subspace of $H_{b,l}$ and satisfy:*

(i) $\|D\Phi(p)\|_{\mathcal{L}(P_N V; Q_N V)} \leq b_1, \quad b_1 > 0, \quad \forall\, p \in P_N V,$

(ii) $\|D\Phi(p_1) - D\Phi(p_2)\|_{\mathcal{L}(P_N V; Q_N V)} \leq l_1\|p_1 - p_2\|, \quad \forall\, p_1, p_2 \in P_N V.$

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Note that $H_{b,l}^1$ is complete [12] with the metric

$$\text{dist}(\Phi, \Phi') = \sup_{\boldsymbol{p} \in P_N V} \|\Phi(\boldsymbol{p}) - \Phi'(\boldsymbol{p})\|.$$

In the Lemma 4.2 we established the global boundedness and Lipschitz continuity for the modified nonlinear term $F(\boldsymbol{u})$. We will now show similar properties for the Fréchet derivative $[DF](\boldsymbol{u})$.

Lemma 4.8 *For $\Phi \in H_{b,l}$ and $\boldsymbol{p}_1, \boldsymbol{p}_2 \in P_N V$, we have*

$$\text{(i)} \quad |[DF](\boldsymbol{u})\,\boldsymbol{h}| \leq K_1 \|\boldsymbol{h}\|, \quad \text{for fixed } \boldsymbol{u} \in V; \; \forall\, \boldsymbol{h} \in V. \tag{4.46}$$

$$\text{(ii)} \quad |[DF](\boldsymbol{u}_1)\,\boldsymbol{h}_1 - [DF](\boldsymbol{u}_2)\,\boldsymbol{h}_2|$$

$$\leq (1 + l)(K_2 \|\boldsymbol{h}_1\| + c_0 \|\boldsymbol{h}_2\|) \|\boldsymbol{p}_1 - \boldsymbol{p}_2\| + K_1 \|\boldsymbol{h}_1 - \boldsymbol{h}_2\|,$$

$$\forall\, \boldsymbol{u}_1, \boldsymbol{u}_2 \in V; \; \forall\, \boldsymbol{h}_1, \boldsymbol{h}_2 \in V. \tag{4.47}$$

Proof: (i) Note that the Fréchet derivative $[DR](\boldsymbol{u})$ of $R(\boldsymbol{u})$ is

$$[DR](\boldsymbol{u})\,\boldsymbol{h} = \nu A_1 \boldsymbol{h} + B(\boldsymbol{u}, \boldsymbol{h}) + B(\boldsymbol{h}, \boldsymbol{u}) \quad \text{for fixed } \boldsymbol{u} \in V. \tag{4.48}$$

From (4.18) and (4.19), we get

$$|[DR](\boldsymbol{u})\,\boldsymbol{h}| \leq \nu |A_1 \boldsymbol{h}| + |B(\boldsymbol{u}, \boldsymbol{h})| + |B(\boldsymbol{h}, \boldsymbol{u})|$$

$$\leq (\nu + 2c_0 \|\boldsymbol{u}\|) \|\boldsymbol{h}\|.$$

This gives

$$|[DF](\boldsymbol{u})\,\boldsymbol{h}| \leq \theta_{r_1}(\|\boldsymbol{u}\|) \, |[DR](\boldsymbol{u})\,\boldsymbol{h}|$$

$$\leq (\nu + 4c_0 r_1) \|\boldsymbol{h}\|, \quad \forall\, \boldsymbol{h} \in V.$$

Thus for fixed $\boldsymbol{u} \in V$, $\|[DF](\boldsymbol{u})\|_{\mathcal{L}(V;H)} \leq K_1$ with $K_1 = \nu + 4c_0 r_1$.

(ii) For any $\boldsymbol{h}_1, \boldsymbol{h}_2 \in V$, We have from (4.48),

$$[DR](\boldsymbol{u}_1)\,\boldsymbol{h}_1 - [DR](\boldsymbol{u}_2)\,\boldsymbol{h}_2 = \nu A_1(\boldsymbol{h}_1 - \boldsymbol{h}_2) + B(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{h}_1)$$

$$+ B(\boldsymbol{u}_2, \boldsymbol{h}_1 - \boldsymbol{h}_2) + B(\boldsymbol{h}_1 - \boldsymbol{h}_2, \boldsymbol{u}_1) + B(\boldsymbol{h}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2).$$

It follows from (4.18), (4.19) that

$$| [DR](u_1)\,h_1 - [DR](u_2)\,h_2 |$$

$$\leq (\nu + c_0\|u_2\| + c_0\|u_1\|)\,\|h_1 - h_2\| + c_0(\|h_1\| + \|h_2\|)\,\|u_1 - u_2\|.$$

Now, let us consider

$$[DF](u_1)\,h_1 - [DF](u_2)\,h_2 = \theta_{r_1}(\|u_1\|)[DR](u_1)\,h_1 - \theta_{r_1}(\|u_2\|)[DR](u_2)\,h_2,$$

for the following three cases. First suppose that $\|u_1\|, \|u_2\| \geq 2r_1$, then we immediately have $[DF](u_1)\,h_1 - [DF](u_2)\,h_2 = 0$. If $\|u_1\| < 2r_1 \leq \|u_2\|$, then $\theta_{r_1}(\|u_2\|) = 0$. This gives

$$| [DF](u_1)\,h_1 - [DF](u_2)\,h_2 | \leq | \theta_{r_1}(\|u_1\|) - \theta_{r_1}(\|u_2\|) |\,| [DR](u_1)\,h_1 |$$

$$\leq 2r_1^{-1} K_1 \|u_1 - u_2\|\,\|h_1\|.$$

For the case $\|u_1\|, \|u_2\| < 2r_1$, we have

$$| [DF](u_1)\,h_1 - [DF](u_2)\,h_2 |$$

$$\leq | \theta_{r_1}(\|u_1\|) - \theta_{r_1}(\|u_2\|) |\,| [DR](u_1)\,h_1 |$$

$$+ \theta_{r_1}(\|u_2\|)\,| [DR](u_1)\,h_1 - [DR](u_2)\,h_2 |$$

$$\leq \left[(2r_1^{-1} K_1 + c_0)\|h_1\| + c_0\|h_2\| \right]\,\|u_1 - u_2\| + K_1 \|h_1 - h_2\|.$$

Combining these results gives,

$$| [DF](u_1)\,h_1 - [DF](u_2)\,h_2 |$$

$$\leq (K_2\|h_1\| + c_0\|h_2\|)\,\|u_1 - u_2\| + K_1 \|h_1 - h_2\|, \tag{4.49}$$

with $K_2 = 2r_1^{-1} K_1 + c_0$.

Writing $u_1 - u_2 = p_1 - p_2 + [\Phi(p_1) - \Phi(p_2)]$ for $\Phi \in H_{b,l}$, we get $\|u_1 - u_2\| \leq (1+l)\|p_1 - p_2\|$. Hence, for any $h_1, h_2 \in V$, (4.49) becomes

$$| [DF](u_1)\,h_1 - [DF](u_2)\,h_2 |$$

$$\leq (1+l)(K_2\|h_1\| + c_0\|h_2\|)\,\|p_1 - p_2\| + K_1 \|h_1 - h_2\|.$$

In order to prove the existence of finite dimensional C^1 variety $\mathcal{M}$ using contraction mapping theorem, we establish the

Lemma 4.9 *For $\Phi \in H_{b,l}$, if $\mu_{N+1}^{1/2} - \mu_N^{1/2} \geq \epsilon^{-1} K_7$, $\mu_{N+1} - \mu_N \geq \epsilon^{-2} K_6$ and $l \leq 2$ then T maps $H_{b,l}^1$ into itself:*

$$T : H_{b,l}^1 \longmapsto H_{b,l}^1,$$

where $K_7 = \max\{d_3, K_3, K_5\}$ and $K_6 = K_1 K_4 (1 + b_1)$.

Proof: Let us first show that $D(T\Phi)(p_0)$ is globally bounded in $\mathcal{L}(P_N V; Q_N V)$ with respect to $p_0 \in P_N V$.

Lemma 4.10 *If $\Phi \in H_{b,l}^1$ and $p_0 \in P_N V$, then $[D(T\Phi)](p_0) \in \mathcal{L}(P_N V; Q_N V)$. Moreover, if $\delta_{1_N} = \epsilon(\mu_{N+1} - \mu_N) - K_1(1 + b_1)\mu_N^{1/2} > 0$ then we have*

$$\|[D(T\Phi)](p_0)\|_{\mathcal{L}(P_N V; Q_N V)} \leq M_1,$$

where $M_1 = K_1(1 + b_1)\mu_N^{-1/2}\left[\epsilon^{-1} + (\epsilon - \sigma_N \xi_{1_N})^{-1}\right] e^{-1/2} \exp \frac{\sigma_N \xi_{1_N}}{2\epsilon}$, and $\xi_{1_N} = \epsilon + K_1(1 + b_1)\mu_N^{-1/2}$.

Proof: From the definition of $T\Phi(p_0)$ in (4.26), we have $\forall h \in P_N V$

$$[D(T\Phi)](p_0)h = -\int_{-\infty}^0 Q_N S(-\tau)Q_N[DF](u) \circ (I + [D\Phi](p)) \circ [Dp](p_0)h \, d\tau.$$

Note that from Lemma 4.5, we have $(T\Phi)(p_0) \in Q_N V$. This means $(T\Phi)(\cdot)$ is a nonlinear mapping from $P_N V$ into $Q_N V$. Hence the Fréchet derivative, $[D(T\Phi)](p_0) \in \mathcal{L}(P_N V; Q_N V)$ for fixed $p_0 \in P_N V$. By the definition of $[D(T\Phi)](p_0)$ given above, such result can be shown as follows: Note that $p(t) \equiv p(t; \Phi, p_0)$ implies $p(\cdot) : P_N V \mapsto P_N V$. Thus Fréchet derivative of $p(t)$ at p_0 is $[Dp](p_0) \in \mathcal{L}(P_N V; P_N V)$. Now note that $\Phi(\cdot) : P_N V \mapsto Q_N V$ implies $[D\Phi](p) \in \mathcal{L}(P_N V; Q_N V)$. We get

$$[Dp](p_0) + [D\Phi](p) \circ [Dp](p_0) \in \mathcal{L}(P_N V; V).$$

Since $F(\cdot)$ maps V into H, we obtain $[DF](u) \in \mathcal{L}(V; H)$. Consequently, we have $Q_N[DF](u) \in \mathcal{L}(V; Q_N H)$. Combining all the above results :

$$Q_N[DF](u) \circ (I + [D\Phi](p)) \circ [Dp](p_0) \in \mathcal{L}(P_N V; Q_N H).$$

Recalling the properties of the holomorphic semigroup $S(-\tau)$ described in section 3.1, we have $Q_N S(-\tau)(\cdot) : Q_N H \mapsto Q_N V$. Hence $D[T\Phi](p_0) \in \mathcal{L}(P_N V; Q_N V)$.

Let us now set $\psi = (I + [D\Phi](p)) \circ [Dp](p_0) h \in V$, then we get

$$\|[D(T\Phi)](p_0) h\| \leq \int_{-\infty}^{0} \|Q_N S(-\tau)\|_{\mathcal{L}(Q_N H; Q_N V)} |Q_N[DF](u)\psi| \, d\tau. \qquad (4.50)$$

First we need to estimate $|Q_N[DF](u)\psi|$. Since $\psi \in V$, we have from (4.46) and the estimate $\|[D\Phi](p)\|_{\mathcal{L}(P_N V; Q_N V)} \leq b_1$,

$$|[DF](u)\psi| \leq K_1 \|(I + [D\Phi](p)) \circ [Dp](p_0) h\|$$
$$\leq K_1 (1 + b_1) \|[Dp](p_0) h\|.$$

Let us consider $g = [Dp](p_0) h \in P_N V$, for $h \in P_N V$. Taking the Fréchet derivative with respect to p_0 in the following evolution equation

$$\begin{cases} p_t + \epsilon A p + P_N F(u) = 0 \\ p(t) \equiv p(t; \Phi, p_0), \end{cases}$$

we obtain

$$\begin{cases} g_t + \epsilon A g + P_N[DF](u)\psi = 0 \\ g(0) = h. \end{cases} \qquad (4.51)$$

Taking inner product with Ag for (4.51) to get

$$\frac{1}{2} \frac{d}{dt} \|g\|^2 + \epsilon |Ag|^2 \leq K_1 (1 + b_1) \|g\| \, |Ag|.$$

Then, by using the fact that $|Ag|^2 \leq \mu_N \|g\|^2$ we obtain

$$\|g\| \frac{d}{dt} \|g\| \geq -[\epsilon \mu_N + K_1 (1 + b_1) \mu_N^{1/2}] \|g\|^2.$$

This implies

$$\frac{d}{dt} \|g\| + \mu_N \xi_{1_N} \|g\| \geq 0, \quad \text{where } \xi_{1_N} = \epsilon + K_1 (1 + b_1) \mu_N^{-1/2}.$$

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Integrating from τ to 0 gives

$$\|g(\tau)\| \leq \|g(0)\| \exp\{-\mu_N \xi_{1_N} \tau\}$$

$$\leq \|h\| \exp\{-\mu_N \xi_{1_N} \tau\}, \quad \forall \tau \leq 0. \tag{4.52}$$

Consequently, we have

$$|[DF](u)\psi| \leq K_1(1 + b_1)\|h\| \exp\{-\mu_N \xi_{1_N} \tau\}, \quad \forall \tau \leq 0. \tag{4.53}$$

By applying result (4.53) in (4.50) we obtain

$$\|[D(T\Phi)](p_0)h\| \leq K_1(1 + b_1)\|h\| \int_{-\infty}^{0} \|Q_N S(-\tau)\|_{\mathcal{L}(Q_N H; Q_N V)} e^{-\mu_N \xi_{1_N} \tau} d\tau$$

$$\leq K_1(1 + b_1)\|h\| \left\{ \mu_{N+1}^{1/2} \int_{-\infty}^{-1/2\epsilon\mu_{N+1}} e^{\delta_{1_N} \tau} d\tau \right.$$

$$\left. + \left(\frac{1}{2\epsilon e}\right)^{1/2} \int_{-1/2\epsilon\mu_{N+1}}^{0} |\tau|^{-1/2} e^{-\mu_N \xi_{1_N} \tau} d\tau \right\}.$$

Note that here we have similar estimates as in (4.35), (4.36). i.e.

$$\mu_{N+1}^{1/2} \int_{-\infty}^{-1/2\epsilon\mu_{N+1}} e^{\delta_{1_N} \tau} d\tau \leq e^{-1/2} \mu_{N+1}^{-1/2} (\epsilon - \sigma_N \xi_{1_N})^{-1} e^{\frac{\sigma_N \xi_{1_N}}{2\epsilon}},$$

with $\delta_{1_N} = \epsilon(\mu_{N+1} - \mu_N) - K_1(1 + b_1)\mu_N^{1/2}$, $\sigma_N = \mu_N/\mu_{N+1}$ and

$$\left(\frac{1}{2\epsilon e}\right)^{1/2} \int_{-1/2\epsilon\mu_{N+1}}^{0} |\tau|^{-1/2} e^{-\mu_N \xi_{1_N} \tau} d\tau \leq \epsilon^{-1} e^{-1/2} \mu_{N+1}^{-1/2} \exp^{\frac{\sigma_N \xi_{1_N}}{2\epsilon}}.$$

We then have

$$\|[D(T\Phi)](p_0)h\| \leq M_1 \|h\|, \quad \forall h \in P_N V,$$

with $M_1 = K_1(1 + b_1)\mu_N^{-1/2} \left[\epsilon^{-1} + (\epsilon - \sigma_N \xi_{1_N})^{-1}\right] e^{-1/2} \exp \frac{\sigma_N \xi_{1_N}}{2\epsilon}$. Thus

$$\|[D(T\Phi)](p_0)\|_{\mathcal{L}(P_N V; Q_N V)} \leq M_1. \quad \_$$

$$\square$$

Let us now prove the global Lipschitz continuity of map $[D(T\Phi)](\cdot)$. Note that here $[D(T\Phi)](\cdot)$ defines an operator valued map $P_N V \mapsto \mathcal{L}(P_N V; Q_N V)$.

Lemma 4.11 *If* $\delta'_{1_N} = \epsilon\mu_{N+1} - \mu_N(\xi_{1_N} + \xi_N) > 0$ *then for* $\Phi \in H^1_{b,l}$ *and* $p_{01}, p_{02} \in P_N V$ *we have*

$$\| [D(T\Phi)](p_{01}) - [D(T\Phi)](p_{02})\|_{\mathcal{L}(P_N V; Q_N V)} \le l'_1 \|p_{01} - p_{02}\|,$$

with

$$l'_1 = c_3 \left[1 + K_1(1 + b_1)\epsilon^{-1}\mu_N^{-1/2} \right]$$
$$\times \left\{ \mu_{N+1}^{-1/2} \left[\epsilon^{-1} + (\epsilon - \sigma_N(\xi_{1_N} + \xi_N))^{-1} \right] e^{-1/2} \exp \frac{\sigma_N(\xi_{1_N} + \xi_N)}{2\epsilon} \right\}.$$

Proof: Let Φ be a fixed element in $H^1_{b,l}$. Let $p_1(t)$ and $p_2(t)$ be two different solutions of (4.15) with $p_1(0) = p_{01}$ and $p_2(0) = p_{02}$ respectively. Let us first consider the term $|[DF](u_1)\psi_1 - [DF](u_2)\psi_2|$, from (4.47) we have

$$|[DF](u_1)\,\psi_1 - [DF](u_2)\,\psi_2|$$
$$\le (1 + l)(K_2\|\psi_1\| + c_0\|\psi_2\|)\|p_1 - p_2\| + K_1\|\psi_1 - \psi_2\|, \qquad (4.54)$$

with $\psi_1 = (I + [D\Phi](p_1)) \circ [Dp_1](p_{01})h \in V$ and $\psi_2 = (I + [D\Phi](p_2)) \circ [Dp_2](p_{02})h \in V$, $\forall\ h \in P_N V$. By setting $g_1 = [Dp_1](p_{01})h \in P_N V$ and $g_2 = [Dp_2](p_{02})h \in P_N V$, we immediately have

$$\|\psi_1\| \le (1 + b_1)\|g_1\| \qquad (4.55)$$
$$\text{and} \quad \|\psi_2\| \le (1 + b_1)\|g_2\|. \qquad (4.56)$$

Since $\psi_1 - \psi_2 = (g_1 - g_2) + ([D\Phi](p_1)g_1 - [D\Phi](p_2)g_2)$,

$$\|\psi_1 - \psi_2\| \le \|g_1 - g_2\| + \| [D\Phi](p_1)g_1 - [D\Phi](p_2)g_2\|,$$

where

$$\| [D\Phi](p_1)g_1 - [D\Phi](p_2)g_2\|$$
$$\le \| [D\Phi](p_1)(g_1 - g_2)\| + \| ([D\Phi](p_1) - [D\Phi](p_2))g_2\|$$
$$\le b_1\|g_1 - g_2\| + l_1\|p_1 - p_2\|\,\|g_2\|.$$

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This gives

$$\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\| \le (1 + b_1)\|\boldsymbol{g}_1 - \boldsymbol{g}_2\| + l_1\|\boldsymbol{p}_1 - \boldsymbol{p}_2\| \, \|\boldsymbol{g}_2\|. \tag{4.57}$$

By applying (4.55), (4.56) and substituting (4.57) into (4.54) we get

$$| \, [DF](\boldsymbol{u}_1) \, \boldsymbol{\psi}_1 - [DF](\boldsymbol{u}_2) \, \boldsymbol{\psi}_2 |$$

$$\le (c_1\|\boldsymbol{g}_1\| + c_2\|\boldsymbol{g}_2\|) \, \|\boldsymbol{p}_1 - \boldsymbol{p}_2\| + K_1(1 + b_1)\|\boldsymbol{g}_1 - \boldsymbol{g}_2\|,$$

where $c_1 = K_2(1 + l)(1 + b_1)$, $c_2 = c_0(1 + l)(1 + b_1) + K_1 l_1$. From (4.52), note that we also have the following estimates:

$$\|\boldsymbol{g}_1(\tau)\| \le \exp\{-\mu_N \xi_{1_N} \tau\} \, \|\boldsymbol{h}\|, \quad \forall \, \tau \le 0;$$

$$\|\boldsymbol{g}_2(\tau)\| \le \exp\{-\mu_N \xi_{1_N} \tau\} \, \|\boldsymbol{h}\|, \quad \forall \, \tau \le 0;$$

$$\|\boldsymbol{p}_1(\tau) - \boldsymbol{p}_1(\tau)\| \le \exp\{-\mu_N \xi_N \tau\}\|\boldsymbol{p}_{01} - \boldsymbol{p}_{02}\| \quad \forall \, \tau \le 0.$$

Now, we need to estimate $\|\boldsymbol{g}_1 - \boldsymbol{g}_2\|$. Let us consider the following two evolutionary problems for $\boldsymbol{g}_1(t)$ and $\boldsymbol{g}_2(t)$ respectively:

$$\begin{cases} \boldsymbol{g}_{1_t} + \epsilon A \boldsymbol{g}_1 + P_N[DF](\boldsymbol{u}_1)\boldsymbol{\psi}_1 = 0 \\ \boldsymbol{g}_1(0) = \boldsymbol{h}, \quad \forall \, \boldsymbol{h} \in P_N V, \end{cases} \tag{4.58}$$

and

$$\begin{cases} \boldsymbol{g}_{2_t} + \epsilon A \boldsymbol{g}_2 + P_N[DF](\boldsymbol{u}_2)\boldsymbol{\psi}_2 = 0 \\ \boldsymbol{g}_2(0) = \boldsymbol{h}, \quad \forall \, \boldsymbol{h} \in P_N V, \end{cases} \tag{4.59}$$

with $\boldsymbol{u}_1 = \boldsymbol{p}_1 + \Phi(\boldsymbol{p}_1)$ and $\boldsymbol{u}_2 = \boldsymbol{p}_2 + \Phi(\boldsymbol{p}_2)$. By subtracting (4.59) from (4.58), we get

$$\begin{cases} \boldsymbol{g}_t + \epsilon A \boldsymbol{g} + P_N[DF](\boldsymbol{u}_1)\boldsymbol{\psi}_1 - P_N[DF](\boldsymbol{u}_2)\boldsymbol{\psi}_2 = 0 \\ \boldsymbol{g}(t) = \boldsymbol{g}_1(t) - \boldsymbol{g}_2(t), \quad \boldsymbol{g}(0) = 0. \end{cases} \tag{4.60}$$

Taking inner product of $A\boldsymbol{g}$ with (4.60) to get

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{g}\|^2 + \epsilon|A\boldsymbol{g}|^2$$

$$\le | \, [DF](\boldsymbol{u}_1) \, \boldsymbol{\psi}_1 - [DF](\boldsymbol{u}_2) \, \boldsymbol{\psi}_2 | \, |A\boldsymbol{g}|$$

$$\le \left[c_3\|\boldsymbol{p}_{01} - \boldsymbol{p}_{02}\| \, \|\boldsymbol{h}\| e^{-\mu_N(\xi_{1_N} + \xi_N)t} + K_1(1 + b_1)\|\boldsymbol{g}\|\right] \, \mu_N^{1/2}\|\boldsymbol{g}\|,$$

which implies

$$\frac{d}{dt}\|g\| + \mu_N \xi_{1_N}\|g\| \geq -c_3\, \mu_N^{1/2} e^{-\mu_N(\xi_{1_N}+\xi_N)t}\|p_{01} - p_{02}\|\, \|h\|,$$

with $c_3 = c_1 + c_2$. This leads to

$$\|g_1(\tau) - g_2(\tau)\| \leq \frac{c_3}{\epsilon}\mu_N^{-1/2}\|p_{01} - p_{02}\| \exp\{-\mu_N(\xi_{1_N} + \xi_N)\tau\}\|h\| \quad \forall\, \tau \leq 0.$$

Hence,

$$\left|\,[DF](u_1)\,\psi_1 - [DF](u_2)\,\psi_2\right|$$
$$\leq c_3\left[1 + K_1(1 + b_1)\epsilon^{-1}\mu_N^{-1/2}\right]e^{-\mu_N(\xi_{1_N}+\xi_N)t}\|p_{01} - p_{02}\|\,\|h\|. \qquad (4.61)$$

From (4.61), we obtain

$$\|\,[D(T\Phi)](p_{01})h - [D(T\Phi)](p_{02})h\|$$
$$\leq \int_{-\infty}^0 \|Q_N S(-\tau)\|_{\mathcal{L}(Q_N H; Q_N V)}|\,[DF](u_1)\,\psi_1 - [DF](u_2)\,\psi_2|$$
$$\leq c_3\left[1 + K_1(1 + b_1)\epsilon^{-1}\mu_N^{-1/2}\right]\|p_{01} - p_{02}\|\,\|h\|$$
$$\times \int_{-\infty}^0 \|Q_N S(-\tau)\|_{\mathcal{L}(Q_N H; Q_N V)}e^{-\mu_N(\xi_{1_N}+\xi_N)\tau}\, d\tau,$$

where

$$\int_{-\infty}^0 \|Q_N S(-\tau)\|_{\mathcal{L}(Q_N H; Q_N V)}e^{-\mu_N(\xi_{1_N}+\xi_N)\tau}\, d\tau$$
$$\leq \mu_{N+1}^{-1/2}\left[\epsilon^{-1} + (\epsilon - \sigma_N(\xi_{1_N} + \xi_N))^{-1}\right]e^{-1/2}\exp\frac{\sigma_N(\xi_{1_N}+\xi_N)}{2\epsilon}$$

with $\delta'_{1_N} = \epsilon\mu_{N+1} - \mu_N(\xi_{1_N} + \xi_N) > 0$. Finally, we get

$$\|\,[D(T\Phi)](p_{01})h - [D(T\Phi)](p_{02})h\| \leq l'_1\|p_{01} - p_{02}\|\,\|h\|, \quad \forall\, h \in P_N V,$$

with

$$l'_1 = c_3\left[1 + K_1(1 + b_1)\epsilon^{-1}\mu_N^{-1/2}\right]$$
$$\times \left\{\mu_{N+1}^{-1/2}\left[\epsilon^{-1} + (\epsilon - \sigma_N(\xi_{1_N} + \xi_N))^{-1}\right]e^{-1/2}\exp\frac{\sigma_N(\xi_{1_N}+\xi_N)}{2\epsilon}\right\}.$$

Thus,

$$\|\,[D(T\Phi)](p_{01}) - [D(T\Phi)](p_{02})\|_{\mathcal{L}(P_N V; Q_N V)} \leq l'_1\|p_{01} - p_{02}\|.$$

Now in order for the transform T to map $H^1_{b,l}$ into itself, along with the hypotheses in Lemma 4.3 we must also have $M_1 \leq b_1$ and $l'_1 \leq l_1$. An elementary calculation as before show that a sufficient condition for $M_1 \leq b_1$ is

$$\mu_{N+1}^{1/2} - \mu_N^{1/2} \geq \epsilon^{-1} K_3, \quad \text{with } K_3 = 2K_1(1+b_1)b_1^{-1}, \qquad (4.62)$$

and for $l'_1 \leq l_1$ is

$$\begin{cases} \mu_{N+1}^{1/2} - \mu_N^{1/2} \geq \epsilon^{-1} K_5, \text{ with } K_5 = \max\{K_4, K_1(1+b_1) + d_2(1+l)\} \\ \\ \mu_{N+1} - \mu_N \geq \epsilon^{-2} K_6, \text{ with } K_6 = K_1 K_4(1+b_1), \ K_4 = 4c_3 l_1^{-1}. \end{cases} \qquad (4.63)$$

Combining above results (4.62), (4.63) with the hypotheses in Lemma 4.3 we get

$$\begin{cases} \mu_{N+1}^{1/2} - \mu_N^{1/2} \geq \epsilon^{-1} K_7, \quad \text{with } K_7 = \max\{d_3, K_3, K_5\} \\ \\ \mu_{N+1} - \mu_N \geq \epsilon^{-2} K_6. \end{cases}$$

In previous section, we have proven that T is a strict contraction map on $H_{b,l}$. Since $H^1_{b,l}$ is a closed subspace of $H_{b,l}$, this immediately implies that T is also a strict contraction on $H^1_{b,l}$. Together with the Lemmas proved above, we can conclude that there exists a unique fixed point $\Phi^* \in H^1_{b,l}$ such that $T\Phi^* = \Phi^*$ by the contraction mapping theorem. Hence we have the following theorem:

Theorem 4.3 *Let $\Phi \in H^1_{b,l}$ and with the hypotheses be given in Theorem 4.1. There exist constants d_4, K_6, K_7 depending only on l, l_1, b_1, r_1, ν, f and Ω such that the following conditions are satisfied:*

(i) $N \geq N^*$;

(ii) $\mu_{N+1}^{1/2} \geq \epsilon^{-1} d_4$;

(iii) $\mu_{N+1}^{1/2} - \mu_N^{1/2} \geq \epsilon^{-1} K_7$;

(iv) $\mu_{N+1} - \mu_N \geq \epsilon^{-2} K_6$.

Then the tranform T has a fixed point $\Phi \in H^1_{b,l}$. The finite dimensional C^1 manifold $\mathcal{M}$ defined by the graph of Φ is an inertial manifold for (4.12).

$\square$

Let us now compare the spectral gap conditions for Theorem 4.1 and Theorem 4.3. Notice that for a fixed $\epsilon > 0$ there exist infinitely many choices of N such that the conditions in Theorem 4.1 are satisfied. Let us assume now that N_0 is the smallest number for which the conditions in Theorem 4.1 are satisfied. Now, for the existence of a C^1 inertial manifold, additional conditions (iii) and (iv) in Theorem 4.3 are needed. Note that condition (iv) in Theorem 4.3 can be written as

$$\left(\mu_{N+1}^{1/2} - \mu_N^{1/2}\right) \left(\mu_{N+1}^{1/2} + \mu_N^{1/2}\right) \geq \frac{1}{\epsilon^2} K_6.$$

Since $\mu_{N+1} \geq \mu_N$, we obtain

$$\mu_{N+1}^{1/2} - \mu_N^{1/2} \geq \frac{1}{\epsilon} \left(\frac{K_6}{2\,\epsilon\,\mu_{N+1}^{1/2}}\right).$$

For a fixed $\epsilon > 0$ and every $N > N_0$, we have

$$\begin{aligned}
\mu_{N+1}^{1/2} - \mu_N^{1/2} &\geq \frac{1}{\epsilon}\left(\frac{K_6}{2 d_4}\right) = \frac{1}{\epsilon}\left(\frac{K_6}{2\epsilon}\frac{\epsilon}{d_4}\right) \\
&\geq \frac{1}{\epsilon}\left(\frac{K_6}{2\epsilon}\frac{1}{\mu_{N_0+1}^{1/2}}\right) \\
&> \frac{1}{\epsilon}\left(\frac{K_6}{2\epsilon}\frac{1}{\mu_{N+1}^{1/2}}\right).
\end{aligned}$$

Hence, provided that $N > N_0$, condition (iii) and (iv) can be combined into one condition. i.e.

$$\mu_{N+1}^{1/2} - \mu_N^{1/2} \geq \frac{1}{\epsilon} K_8 \quad \text{with } K_8 = \max\{\frac{K_6}{2 d_4}, K_7\}.$$

This means for $N > N_0$, a sufficient condition for the existence of a C^1 inertial manifold is

$$\mu_{N+1}^{1/2} - \mu_N^{1/2} \geq \epsilon^{-1} K_8.$$

Such result implies that the higher dimensional inertial manifolds are automatically C^1.

Appendix A

Time Analyticity of Strong Solution

In this section we will show the time analyticity of the strong solution using a method introduced in [21] for the conventional Navier-Stokes equations. Let us first recall an existence theorem provided in [15] for the evolutionary equation of (4.11).

Theorem A.1 *Let $n \leq 6$ and $u_0 \in V$, $f \in L^\infty(0, T; H)$ be given. Then there exists a unique solution of problem (4.11), which satisfies $u \in L^2(0, T; D(A)) \cap C([0, T]; V)$ and $u' \in L^2(0, T; H)$.*

$\square$

Let us denote $\mathbf{C}$ as the complex plane and $H_{\mathbf{C}}, V_{\mathbf{C}}, D(A_{\mathbf{C}})$ be respectively the complexification of the spaces $H, V, D(A)$.

Theorem A.2 *Let f be an $H_{\mathbf{C}}$-holomorphic function on D (a neighborhood of the positive real axis $\mathbf{R}^+$) and is bounded from D into $H_{\mathbf{C}}$. Then, the unique solution u in Theorem A.1 is an $H_{\mathbf{C}}$-holomorphic function in a neighborhood of the positive real axis.*

Proof : By considering projection of (4.11) in $P_N H_{\mathbf{C}}$ we get the complex differential system

$$\frac{du_N(\varsigma)}{d\varsigma} + \epsilon A u_N(\varsigma) + \nu P_N A_1 u_N(\varsigma) + P_N B(u_N(\varsigma), u_N(\varsigma)) = P_N f \qquad (A.1)$$

$$u_N(0) = P_N u_0. \qquad (A.2)$$

The above system is equivalent to a finite system of complex ordinary differential equations. Such a system has a unique solution $u_N(\varsigma)$ which is holomorphic in a neighborhood of the origin [3, Theorem 8.1, Chapter 1]. Let us now obtain an estimate on the size of this neighborhood in the ς-plane.

By taking inner product of (A.1) with $Au_N(\varsigma)$, we get

$$\frac{1}{2}\frac{d}{ds}\|u_N(se^{i\theta})\|^2 + \epsilon\cos\theta|Au_N(se^{i\theta})|^2 + \nu\cos\theta|\nabla A_2 u_N(se^{i\theta})|^2$$
$$= -\text{Re}\{e^{i\theta}(B(u_N(se^{i\theta}), u_N(se^{i\theta})), Au_N(se^{i\theta}))\} + \text{Re}\{e^{i\theta}(f, Au_N(se^{i\theta}))\}. \quad \text{(A.3)}$$

The right hand side of above equation has following estimates:

$$2\left|(f, Au_N(se^{i\theta}))\right| \leq 2|f||Au_N(se^{i\theta})|$$
$$\leq \frac{\epsilon\cos\theta}{2}|Au_N|^2 + \frac{2}{\epsilon\cos\theta}|f|^2.$$

The estimate for trilinear term of real case in (2.10) can be extended to the complex case

$$|b(u, v, w)| \leq c_1'|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w|$$

$$\forall\, u \in V_{\mathbf{C}},\, v \in D(A_{\mathbf{C}})\, w \in H_{\mathbf{C}}.$$

This gives us

$$2\left|(B(u_N(se^{i\theta}), u_N(se^{i\theta})), Au_N(se^{i\theta}))\right|$$
$$\leq 2c_1'|u_N(se^{i\theta})|^{1/2}\|u_N(se^{i\theta})\|\,|Au_N(se^{i\theta})|^{3/2}$$
$$\leq \frac{\epsilon\cos\theta}{2}|Au_N(se^{i\theta})|^2 + \frac{c_2'}{(\cos\theta)^3}|u_N(se^{i\theta})|^2\|u_N(se^{i\theta})\|^4$$
$$\leq \frac{\epsilon\cos\theta}{2}|Au_N(se^{i\theta})|^2 + \frac{c_3'}{(\cos\theta)^3}\|u_N(se^{i\theta})\|^6.$$

Thus equation (A.3) becomes

$$\frac{d}{ds}\|u_N(se^{i\theta})\|^2 + \epsilon\cos\theta|Au_N(se^{i\theta})|^2 + 2\nu\cos\theta|\nabla A_2 u_N(se^{i\theta})|^2$$
$$\leq \frac{2}{\epsilon\cos\theta}|f|^2 + \frac{c_3'}{(\cos\theta)^3}\|u_N(se^{i\theta})\|^6.$$

We can drop positive terms $\epsilon \cos \theta |Au_N(se^{i\theta})|^2$ and $\nu \cos \theta |\nabla A_2 u_N(se^{i\theta})|^2$ to obtain a differential inequality,

$$\frac{d}{ds}\|u_N(se^{i\theta})\|^2 \leq \frac{2}{\epsilon \cos \theta}|f|^2 + \frac{c_3'}{(\cos \theta)^3}\|u_N(se^{i\theta})\|^6$$
$$\leq \frac{c_4'}{(\cos \theta)^3}\left[1 + \|u_N(se^{i\theta})\|^2\right]^3, \quad\quad\quad (A.4)$$

where $c_4' = \max(c_3', 2|f|_\infty/\epsilon)$ and $|f|_\infty = |f|_{L^\infty(0,T;H_{\mathbb{C}})}$. Then, by integrating the differential inequality (A.4) from 0 to s we get

$$\|u_N(se^{i\theta})\|^2 \leq 1 + 2\|u_0\|^2$$

with

$$0 \leq s \leq \alpha(\cos \theta)^3,$$

and

$$\alpha = \frac{3}{8}\frac{1}{c_4'(1 + \|u_0\|^2)^2}.$$

This show that the Galerkin solution $u_N(\varsigma)$ is uniformly bounded in the region $\Delta_D(0)$ of the complex plane $\mathbb{C}$. Here,

$$\Delta_D(0) = D \cap \Delta(0), \quad \Delta(0) = \{\varsigma = se^{i\theta}, 0 \leq s \leq \alpha(\cos \theta)^3, |\theta| < \frac{\pi}{2}\}.$$

Therefore, the domain of holomorphy in time can be extended from the neighborhood of the origin to $\Delta_D(0)$. Moreover, for any $\varsigma = se^{i\theta} \in \Delta_D(0)$, we have

$$\sup_{\varsigma \in \Delta_D(0)} \|u_N(\varsigma)\|^2 \leq k_1, \quad k_1 = k_1(\|u_0\|). \quad\quad\quad (A.5)$$

Let us now consider the passage to the limit of the solution $u_N(\varsigma)$. From the estimate (A.5) and the compactness of the embedding $V_{\mathbb{C}} \subset H_{\mathbb{C}}$, we conclude that for $\varsigma \in \Delta_D(0)$, $\{u_N(\varsigma)\}_{N=1}^\infty$ belongs to a compact set in $H_{\mathbb{C}}$. Hence, by the vector valued version of the Vitali Convergence Theorem [7, Theorem 8.2.1], there is a subsequence $\{u_{N_i}(\varsigma)\}_{i=1}^\infty$ of $\{u_N(\varsigma)\}_{N=1}^\infty$ that uniformly converges to $H_{\mathbb{C}}$-holomorphic function $u^*(\varsigma)$ on every compact subset of $\Delta_D(0)$. Notice that the restriction of $u_N(\varsigma)$ to the positive real axis will concides

with the Galerkin solution $u_N(t)$. Hence, it is also true that the restriction of $u^*(\varsigma)$ to the positive real axis concides with the unique solution $u(t)$ described in Theorem A.1. Therefore, $u^*(\varsigma)$ can be viewed as a holomorphic extension of $u(t)$ to the region $\Delta_D(0)$. This implies $u^*(\varsigma)$ is unique as well in the holomorphic region $\Delta_D(0)$. Due to the uniqueness of $u^*(\varsigma)$ on $\Delta_D(0)$, the entire sequence $\{u_N(\varsigma)\}_{N=1}^{\infty}$ should converge to $u^*(\varsigma)$ in H_C uniformly on every compact subset of $\Delta_D(0)$. We will thus denote $u^*(\varsigma) = u(\varsigma)$.

Let us now consider the inequality in (A.4) for $0 < t_0 < \alpha$ and $s > t_0$. Integrating the differential inequality from t_0 to s we obtain

$$\|u_N(se^{i\theta})\|^2 \le \text{const.}, \quad \text{with } t_0 < s \le t_0 + \alpha(\cos\theta)^3.$$

This implies that $u_N(\varsigma)$ is holomorphic in the region $\Delta_D(t_0)$ of the complex plane C, and

$$\Delta_D(t_0) = D \cap \Delta(t_0), \quad \Delta(t_0) = \{\varsigma = se^{i\theta}, t_0 \le s \le t_0 + \alpha(\cos\theta)^3, |\theta| < \frac{\pi}{2}\}.$$

Iterating this argument, we can conclude that $u(\varsigma)$ is an H_C-holomorphic function in the region Δ of the complex plane C, with

$$\Delta = \bigcup_{t_0 \ge 0} \Delta_D(t_0).$$

That is, $u(\varsigma)$ is an H_C-holomorphic function in a strip containing the positive real axis.

$\square$

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Final Report

16. Abstract

    A practically important regularization of the Navier-Stokes equations have
been analyzed. As a continuation of the previous work, we study in this paper
the structure of the attractors characterizing the solutions. Local as well as
global invariant manifolds have been found. Regularity properties of these
manifolds are analyzed.

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